Stochastic Calculus Assignment 1

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1. Question 6 - Simulating e from a martingale (W1)

Let $\{X_n : n \geq 1\}$ be an i.i.d sequence of uniform random variables over $[0, 1]$ with density $p_{X_n}(x) = \mathbb{1}(0 \leq x \leq 1)$. We can define $S_n \triangleq X_1 + \cdots + X_n$ with $S_0 \triangleq 0$, and further define the stopping time $\tau \triangleq \inf\{n : S_n > 1\}$.

Part a)

Let $f : [0, 2] \to \mathbb{R}$ be a function such that

$$f(x) = \int_0^1 f(x + t)dt + 1 \quad \forall x \in [0, 1] \,. \tag{1.1}$$

We may define a stochastic process, valid for $S_n \leq 1$, as

$$M_n \triangleq f(S_n) + n \quad \text{for} \quad n \geq 0 \,. \tag{1.2}$$

We will show that M_n is a martingale with respect to the natural filtration on S_m, that is $\mathbb{E}[M_{n+1}|\mathcal{F}_n] = M_n$ (which was shown to be equivalent to the condition $\mathbb{E}[M_m|\mathcal{F}_n] = M_n$ for $m > n$ in tutorials) where $\mathcal{F}_n = \sigma(S_1, \ldots, S_n)$. We first note that M_n is clearly $\mathcal{F}_n$-adapted, and measurability follows easily from the compact support and bounded derivatives on $f(x) + n$. For the martingale property, we calculate

$$\begin{aligned}
\mathbb{E}[M_{n+1}|\mathcal{F}_n] &= \mathbb{E}[f(S_n + X_{n+1}) + n + 1|S_n = z] = n + 1 + \mathbb{E}[f(z + X_{n+1})] \\
&= n + 1 + \int_{-\infty}^{\infty} f(z + u)p_{X_n}(u)du = n + 1 + \int_0^1 f(z + u)du \\
&= n + 1 + f(z) - 1 = f(S_n) + n = M_n \,,
\end{aligned} \tag{1.3}$$

thus showing that M_n is a martingale. It is clear that τ is an $\mathcal{F}_n$ stopping time, and so by Proposition 1.1 we know that $\tau \wedge n$ is also an $\mathcal{F}_n$ stopping time, which thus allows us to apply the optional sampling theorem to conclude that $M_{\tau\wedge n}$ is also a martingale with respect to $\mathcal{F}_n$.

Part b)

We can use the above result to prove $\mathbb{E}[\tau] = e$. In particular, since $M_{\tau\wedge n}$ is a martingale we have that

$$\mathbb{E}[M_{\tau\wedge n}|\mathcal{F}_0] = \mathbb{E}[f(S_{\tau\wedge n}) + \tau \wedge n] = M_0 = f(S_0) \,, \tag{1.4}$$

where the second equality is because conditioning on the trivial sigma algebra does not provide any information, and the third equality is due to the martingale property.

We first find an explicit form of $f(x)$ satisfying our above requirements. We note that f is defined on $[0,2]$, but only needs to satisfy the integral equation on $[0,1]$, therefore we may start by defining $f|_{(1,2]} \equiv 0$. We may then perform a substitution on the integral to get

$$\int_0^1 f(x+t)\mathbb{1}(x+t \leq 1)dt = \int_x^{x+1} f(u)\mathbb{1}(u \leq 1)du = \int_x^1 f(u)du\,,$$

and then by the fundamental theorem of calculus we have

$$f'(x) = \frac{d}{dx}\left(-\int_1^x f(u)du\right) = -f(x)\,.$$

We also note that

$$f(1) = \int_1^2 f(u)\mathbb{1}(u \leq 1)du + 1 = 1\,,$$

and so solving the above differential equation with initial condition yields

$$f(x) = Ce^{-x}\mathbb{1}(0 \leq x \leq 1) = e^{1-x}\mathbb{1}(0 \leq x \leq 1)\,. \tag{1.5}$$

To get the desired result, we then want to take $\lim_{n\to\infty} \tau \wedge n = \tau$. To bring this limit inside the expectation we need to apply the dominated convergence theorem, which is valid since we have $f(S_{\tau\wedge n}) + \tau \wedge n \overset{n\to\infty}{\to} f(S_\tau) + \tau$ in the case where τ is finite which happens with probability 1. Further, it is uniformly bounded since $|f(S_{\tau\wedge n}) + \tau \wedge n| \leq |e + \tau|$ and τ is a stopping time that is finite with probability 1. Therefore by the dominated convergence theorem we have

$$\lim_{n\to\infty} \mathbb{E}[f(S_{\tau\wedge n}) + \tau \wedge n] = \mathbb{E}[\lim_{n\to\infty} f(S_{\tau\wedge n}) + \tau \wedge n] = \mathbb{E}[f(S_\tau) + \tau] = \mathbb{E}[\tau] \tag{1.6}$$

where the last equality holds since by definition $S_\tau > 1$, but then we have $f(S_\tau) = 0$. Our calculation in (1.4) then tells us that

$$\mathbb{E}[\tau] = f(S_0) = f(0) = e \tag{1.7}$$

and so we are done. Note that this is clearly a very convenient (even if potentially slow way) of simulating the number e using only a uniform random number generator. □

2. Question 11 - Non-differentiability of Brownian Motion (W1)

Let B be a one-dimensional Brownian motion.

Part a)

Define the random variable

$$X_t \triangleq \begin{cases} tB_{\frac{1}{t}}\,, & \text{if } t > 0 \\ 0 & t = 0 \end{cases} . \tag{2.1}$$

We want to show that $X = \{X_t : t \geq 0\}$ is also Brownian motion, that is, that B satisfies the time inversion property. By definition we have $\mathbb{P}(X_0 = 0) = 1$. For all $t \neq 0$, continuity of paths is immediate since t, $\frac{1}{t}$ and B_t are all continuous functions of t, therefore X_t is just a composite of continuous functions and is thus continuous. For the case $t = 0$, we need to show $\lim_{t\to 0} X_t = \lim_{t\to 0} tB_{1/t} = 0$ almost surely. But this the same as $\lim_{t\to\infty} \frac{B_t}{t} = 0$ (i.e. the law of large numbers for Brownian motion), which we proved in Problem Sheet 4 Q4, thus showing that X_t is continuous almost surely for all $t \geq 0$.

We may then appeal to Proposition 2.2 that states that X is a Brownian motion if and only if it satisfies the properties above, and additionally it is a Gaussian process (i.e. $(X_{t_1}, X_{t_2}, \ldots, X_{t_n})$ is jointly Gaussian for any n and partition $t_1 < \cdots < t_n$) and has mean $\mathbb{E}[X_t] = 0$ and covariance $\mathbb{E}[X_t X_s] = s$ for any $0 \leq s \leq t$. It is clear that distributionally we have $tB_{1/t} \sim t\mathcal{N}(0, 1/t) \sim \mathcal{N}(0, t)$ so $\mathbb{E}[X_t] = 0$.

From the independent incremenet property of Brownian motion, for any $0 \leq s \leq t$ we have

$$\mathbb{E}[B_s B_t] = \mathbb{E}[B_s(B_t - B_s + B_s)] = \mathbb{E}[B_s(B_t - B_s)] + \mathbb{E}[B_s^2] = s\,. \tag{2.2}$$

Therefore we calculate for any $0 \leq s \leq t$, where $\frac{1}{t} \leq \frac{1}{s}$

$$\mathbb{E}[X_s X_t] = \mathbb{E}[stB_{1/s}B_{1/t}] = st\mathbb{E}[B_{1/s}B_{1/t}] = st\frac{1}{t} = s\,, \tag{2.3}$$

so X_t satisfies the covariance property. We further recall that B itself is a Gaussian process, so any random vector $(X_{t_1}, \ldots, X_{t_n})^T = A(B_{1/t_1}, \ldots, B_{1/t_n})^T$ where $A = \mathrm{diag}(t_1, \ldots t_n)$ is a deterministic (constant) matrix, is also a Gaussian process since a linear transformation of a Gaussian random vector is a Gaussian random vector. Therefore X_t is a Brownian motion. $\square$

Part b)

We want to show that with probability one there exist two sequences of positive times $t_n \downarrow 0$ and $s_n \downarrow 0$ such that $B_{s_n} < 0$ and $B_{t_n} > 0$ for all n. Recalling that we define τ_x to be the first time the Brownian motion hits the level x. So for any $n \geq 1$ we may define the sequences

$$t_n := \tau_{1/n} = \inf\{t \geq 0 : B_t = \frac{1}{n}\}\,, \quad s_n := \tau_{-1/n} = \inf\{t \geq 0 : B_t = -\frac{1}{n}\}\,. \tag{2.4}$$

By the symmetry of B we know that $-B$ is also a Brownian motion, thus it only suffices to only consider t_n and all results will similarly apply to s_n. From Lemma 2.3 on the almost surely unboundedness of B, using the continuity in time we know that τ_x is finite almost surely, thus t_n is finite almost surely for each n. We also know that t_n is a monotonically decreasing sequence because for any given path B_t, since $\frac{1}{n+1} < \frac{1}{n}$, using the continuity in t we may apply the intermediate value theorem to see that B_t must hit $\frac{1}{n+1}$ before it hits $\frac{1}{n}$, thus $t_n > t_{n+1}$ for all n. Finally, to see that $t_n \to 0$ almost surely we may use the result from lectures that explicitly calculates the density of τ_x as

$$f_{\tau_x}(t)dt = \frac{x}{\sqrt{2\pi t^3}} e^{-\frac{x^2}{2t}} dt \,. \tag{2.5}$$

Since $t_n > 0$ (so $|t_n - 0| = t_n$) we first want to show that for any $\varepsilon > 0$ we have $\lim_{n\to\infty} \mathbb{P}(t_n < \varepsilon) = 1$. Using the substitution $u = t^{-1/2}$ and then the series expansion of the error function $\text{erf}(x)$ to make the limit easier to see, we calculate

$$\begin{aligned} \mathbb{P}(t_n < \varepsilon) &= \int_0^{\varepsilon} \frac{1}{n\sqrt{2\pi t^3}} e^{-\frac{1}{2n^2 t}} dt = 1 - \text{erf}\left(\frac{1}{n\sqrt{2\varepsilon}}\right) \\ &= 1 - \frac{2}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(2k+1)} \left(\frac{1}{n}\right)^{2k+1} \left(\frac{1}{\sqrt{2\varepsilon}}\right)^{2k+1} \,, \end{aligned} \tag{2.6}$$

and so for any fixed ε we may take the limit $n \to \infty$ of the series to see that every term goes to 0. Therefore we have $\lim_{n\to\infty} \mathbb{P}(t_n < \varepsilon) = 1$. But then since t_n is a monotonically decreasing sequence we must have almost sure convergence, so $t_n \downarrow 0$ almost surely. $\square$

Part c)

We want to show that with probability one $t \mapsto B_t$ is non-differentiable at $t = 0$. Firstly, from Lemma 2.3 we know that almost surely

$$\sup_{m\geq 0} B_m = +\infty \,, \quad \text{and} \quad \inf_{m\geq 0} B_m = -\infty \,. \tag{2.7}$$

So we may define a sequence $a_n = \sup_{m\geq n} B_m$, but then by the strong Markov property we have that

$$a_1 = \sup_{m\geq 1} B_m = \sup_{m\geq 1}(B_m - B_1 + B_1) = \sup_{m\geq 1}(B_m - B_1) + B_1 = \sup_{m'\geq 0}(B_{m'}) + B_1 = +\infty \,. \tag{2.8}$$

Continuing this inductively we see that $a_n = +\infty$ for all n, therefore we conclude that almost surely we have

$$\limsup_{t\to\infty} B_t = \lim_{n\to\infty} a_n = +\infty \,, \quad \text{and similarly} \quad \liminf_{t\to\infty} B_t = -\infty \,. \tag{2.9}$$

Now we turn to differentiability at $t = 0$. Let $t_n \downarrow 0$ and $s_n \downarrow 0$ be defined as above. Then, using the fact that X is also a Brownian motion from part a), we have (where * refers to the upper derivative)

$$B'(0)^* = \limsup_{t_n\to 0} \frac{B(t_n)}{t_n} = \limsup_{t_n\to 0} X\left(\frac{1}{t_n}\right) = \limsup_{t\to\infty} X(t) = +\infty \,, \tag{2.10}$$

but by definition we also have

$$\limsup_{t_n\to 0} \frac{B(t_n)}{t_n} = \limsup_{t_n\to 0} \frac{1}{n t_n} \,. \tag{2.11}$$

Similarly, since s_n is also a sequence of positive times that decrease to 0, we can calculate (where $_*$ refers to the lower derivative)

$$B'(0)_* = \liminf_{s_n\to 0} \frac{B(s_n)}{s_n} = \liminf_{s_n\to 0} \left(-\frac{1}{ns_n}\right) = -\limsup_{s_n\to 0} \frac{1}{ns_n} = -\limsup_{t_n\to 0} \frac{1}{nt_n} = -\infty\,. \tag{2.12}$$

In the second last equality we have used the fact that $-B$ is also a Brownian motion (so $B \sim -B$), so we see that by the symmetry of their definitions we have equivalence in distribution, that is, $t_n \sim s_n$. This says that almost surely we must have $\limsup_{s_n\to 0} \frac{1}{ns_n} = \limsup_{t_n\to 0} \frac{1}{nt_n}$ (in other words, the rate of convergence is not going to be meaningfully different, so their limit will be the same). Thus we have showed that $B'(0)^* = +\infty \neq -\infty = B'(0)_*$ and so B is not differentiable at $t = 0$ since the upper and lower limits do not agree.

Now fix some $t_0 > 0$. We know that the process $Y(t) = B(t + t_0) - B(t_0)$ is also a Brownian motion, which we showed was not differentiable at $t = 0$, which is equivalent to B not being differentiable at t_0 and so we are done. □

Part d)

Let $s < t < u$, we want to calculate $\mathbb{E}[B_t|\sigma(B_s), \sigma(B_u)]$. To begin, we are clearly interested in splitting our interval into more meaningful components, so we can decompose our situation into a Gaussian random variable Z_t (which is Gaussian since B is a Gaussian process) such that

$$(u - s)Z_t = u(B_t - B_s) + s(B_u - B_t) - t(B_u - B_s)\,,$$

which after rearrangement becomes

$$Z_t = B_t - \frac{u-t}{u-s}B_s - \frac{t-s}{u-s}B_u\,. \tag{2.13}$$

It is clear that Z_t is Gaussian since it is just a sum of normally distributed random variables, and further we have

$$\mathbb{E}[Z_t] = \mathbb{E}\left[B_t - \frac{u-t}{u-s}B_s - \frac{t-s}{u-s}B_u\right] = \mathbb{E}[B_t] - \frac{u-t}{u-s}\mathbb{E}[B_s] - \frac{t-s}{u-s}\mathbb{E}[B_u] = 0\,. \tag{2.14}$$

More importantly, Z_t is independent of B_s and B_u since, recalling that it is jointly normal as it is a Gaussian process and the independent increments of Brownian motion, we have

$$\begin{aligned}\mathrm{cov}(Z_t, B_s) = \mathbb{E}[Z_tB_s] &= \mathbb{E}\left[\left(\frac{u}{u-s}(B_t - B_s) + \frac{s}{u-s}(B_u - B_t) - \frac{t}{u-s}(B_u - B_s)\right)B_s\right] \\ &= \frac{u}{u-s}\mathbb{E}[(B_t - B_s)B_s] + \frac{s}{u-s}\mathbb{E}[(B_u - B_t)B_s] - \frac{t}{u-s}\mathbb{E}[(B_u - B_s)B_s] = 0\,.\end{aligned}$$

An identical calculation can be performed for B_u, again using the independent increments of Brownian motion and splitting up the interval accordingly. Rearranging our expression in (2.13), we see that

$$\begin{aligned}\mathbb{E}[B_t|\sigma(B_s), \sigma(B_u)] &= \mathbb{E}[Z_t|\sigma(B_s), \sigma(B_u)] + \mathbb{E}\left[\frac{u-t}{u-s}B_s|\sigma(B_s), \sigma(B_u)\right] + \mathbb{E}\left[\frac{t-s}{u-s}B_u|\sigma(B_s), \sigma(B_u)\right] \\ &= \mathbb{E}[Z_t] + \frac{u-t}{u-s}B_s + \frac{t-s}{u-s}B_u = \frac{u-t}{u-s}B_s + \frac{t-s}{u-s}(B_u - B_s + B_s) \\ &= B_s + \frac{t-s}{u-s}(B_u - B_s)\end{aligned} \tag{2.15}$$

and so we are done. □

3. Question 23 - Brownian motion and the Laplacian (a love story) (W2.5)

Let B^x be a d-dimensional Brownian motion starting at x.

Part a)

Let f be an arbitrary smooth function with compact support $X_d \subset \mathbb{R}^d$. We may define the process

$$M_t \triangleq f(B_t^x) - f(x) - \frac{1}{2}\int_0^t (\Delta f)(B_r^x) dr \,, \tag{3.1}$$

where $\Delta = \Delta_x := \sum_{i=1}^d \frac{\partial^2}{\partial x_i^2}$ is the Laplacian operator on spatial coordinates. We will show M_t is a martingale with respect to the natural filtration on B^x, that is $\mathcal{F}_t = \sigma(\{B_s^x : 0 \le s \le t\})$ for any $t \ge 0$. $\mathcal{F}_t$-adaptedness is clear, and we will assume that measurability is given too since f is smooth on a compact domain. So our main goal is to show $\mathbb{E}[M_t - M_s|\mathcal{F}_s] = 0$ for any $0 \le s < t$. Note that $f(x)$ is a time independent function, which will make our life a bit easier.

We first note that the density of B_t^x (where $x = (x_1, \ldots, x_d)$ is the fixed starting point and $u = (u_1, \ldots, u_d)$ is the differential variable) is given by

$$p(u;t,x) := \frac{1}{(2\pi t)^{d/2}} \exp(-\|u-x\|^2/2t) \,. \tag{3.2}$$

By direct calculation we see that (where dt is the dimension d multiplied by time t, not the differential element $\mathrm{d}t$)

$$\frac{\partial p}{\partial t} = \left(\frac{\|u-x\|^2 - dt}{2t^2} \right) p(u;t,x) \,,$$

and then

$$\frac{\partial p}{\partial u_i} = \frac{\partial p}{\partial u_i} \frac{1}{(2\pi t)^{d/2}} e^{-\frac{(u_i-x_i)^2}{2t}} \prod_{j\neq i} e^{-\frac{(u_j-x_j)^2}{2t}} = -\frac{u_i - x_i}{t} p(u;t,x) \,,$$

which leads to

$$\frac{\partial^2 p}{\partial u_i^2} = \frac{(u_i-x_i)^2 - t}{t^2} p(u;t,x) \,, \quad \text{so} \quad \Delta p = \frac{\|u-x\|^2 - dt}{t^2} p(u;t,x) \,,$$

which leads us to conclude the identity

$$\frac{\partial p}{\partial t} = \frac{1}{2} \Delta p \qquad \text{for all} \quad x \in \mathbb{R}^d, \quad t > 0 \,. \tag{3.3}$$

We now turn to the martingale property. Letting $t = s + h$ for some $h > 0$ in the above martingale property, we have

$$\begin{aligned} M_{s+h} - M_s &= f(B_{s+h}^x) - f(x) - \frac{1}{2}\int_0^{s+h} (\Delta f)(B_r^x) dr - \left(f(B_s^x) - f(x) - \frac{1}{2}\int_0^s (\Delta f)(B_r^x) dr \right) \\ &= f(B_{s+h}^x) - f(B_s^x) - \frac{1}{2}\int_s^{s+h} (\Delta f)(B_r^x) dr \,. \end{aligned} \tag{3.4}$$

Our sigma algebra $\mathcal{F}_s$ gives us all information (i.e. the trajectory) of B_r^x up to time s, so in particular we know that $B_s^x = z$ for some $z \in \mathbb{R}^d$. Taking expectations, where we can bring the expectation inside the integral thanks to Fubini's theorem, we have

$$\begin{aligned}\mathbb{E}[M_{s+h} - M_s|\mathcal{F}_s] &= \mathbb{E}[f(B_{s+h}^x)|\mathcal{F}_s] - \mathbb{E}[f(B_s^x)|\mathcal{F}_s] - \mathbb{E}\left[\int_s^{s+h} \frac{1}{2}(\Delta f)(B_r^x)\mathrm{d}r \,\middle|\, \mathcal{F}_s\right] \\ &= \mathbb{E}[f(B_{s+h}^x)|\mathcal{F}_s] - f(z) - \int_s^{s+h} \mathbb{E}\left[\frac{1}{2}(\Delta f)(B_r^x) \mid \mathcal{F}_s\right] \mathrm{d}r\,.\end{aligned}$$

Thus, using the independent increment, $B_{s+h}^x - B_s^x \sim B_h^0$ and $B_s^0 + z \sim B_s^z$ properties of Brownian motion we have

$$\begin{aligned}\mathbb{E}[f(B_{s+h}^x)|\mathcal{F}_s] &= \mathbb{E}[f\big((B_{s+h}^x - B_s^x) + B_s^x\big)|B_s^x = z] \\ &= \mathbb{E}[f(B_{s+h}^x - B_s^x + z)] = \mathbb{E}[f(B_h^0 + z)] \\ &= \mathbb{E}[f(B_h^z)] = \int_{X_d} f(u)p(u;h,z)\mathrm{d}u\,. \end{aligned} \tag{3.5}$$

We may then calculate, where "$|\mathcal{F}_s$" is notational shorthand to keep in mind that we may use the information provided by $\mathcal{F}_s$ when our calculation requires us to,

$$\begin{aligned}\mathbb{E}\left[\frac{1}{2}(\Delta f)(B_r^x) \mid \mathcal{F}_s\right] &= \int_{X_d} \frac{1}{2}(\Delta f)(u)p(u;r,x)\mathrm{d}u \mid \mathcal{F}_s \\ &= \sum_{i=1}^d \int_{X_d} \frac{1}{2}\frac{\partial^2 f}{\partial u_i^2}(u)p(u;r,x)\mathrm{d}u \mid \mathcal{F}_s\,. \end{aligned} \tag{3.6}$$

Using integration by parts we see that we have

$$\begin{aligned}\int_{X_d} \left(\frac{\partial^2 f}{\partial u_i^2}\right)(u)p(u;r,x)\mathrm{d}u &= \left[\frac{\partial f}{\partial u_i}(u)p(u;r,x)\right]_{\partial X_d} - \int_{X_d} \left(\frac{\partial f}{\partial u_i}\right)(u)\frac{\partial p}{\partial u_i}(u)\mathrm{d}u \\ &= \left[\frac{\partial f}{\partial u_i}(u)p(u;r,x) - f(u)\frac{\partial p}{\partial u_i}(u;r,x)\right]_{\partial X_d} + \int_{X_d} f(u)\left(\frac{\partial^2 p}{\partial u_i^2}\right)(u;r,x)\mathrm{d}u\,. \end{aligned} \tag{3.7}$$

We will then make the simplifying assumption that f is such that the term

$$\sum_{i=1}^d \left[\frac{\partial f}{\partial u_i}(u)p(u;r,x) - f(u)\frac{\partial p}{\partial u_i}(u;r,x)\right]_{\partial X_d} = 0\,, \tag{3.8}$$

which is reasonable to assume since f is compactly supported. Thus, applying (3.7) and (3.3) we have

$$\begin{aligned}\int_{X_d} \frac{1}{2}(\Delta f)(u)p(u;r,x)\mathrm{d}u \mid \mathcal{F}_s &= \int_{X_d} f(u)\frac{1}{2}(\Delta p)(u;r,x)\mathrm{d}u \mid \mathcal{F}_s \\ &= \int_{X_d} f(u)\frac{\partial p}{\partial t}(u;r,x)\mathrm{d}u \mid \mathcal{F}_s\,.\end{aligned}$$

It then follows from Fubini's theorem and our calculation in (3.5) that we have

$$\begin{aligned}
\int_s^{s+h} \mathbb{E}\left[\frac{1}{2}(\Delta f)(B_r^x) \mid \mathcal{F}_s\right] \mathrm{d}r &= \int_s^{s+h} \mathbb{E}\left[\frac{1}{2}(\Delta f)(B_{r-s}^z)\right] \mathrm{d}r \\
&= \lim_{\varepsilon\to 0} \int_{s+\varepsilon}^{s+h} \int_{X_d} \frac{1}{2}(\Delta f)(u) p(u; r-s, z) \mathrm{d}u \mathrm{d}r \\
&= \lim_{\varepsilon\to 0} \int_{X_d} f(u) \left[\int_{s+\varepsilon}^{s+h} \frac{\partial p}{\partial t}(u; r-s, z) \mathrm{d}r\right] \mathrm{d}u \\
&= \lim_{\varepsilon\to 0} \int_{X_d} f(u) \big(p(u; h, z) - p(u; \varepsilon, x)\big) \mathrm{d}u \\
&= \mathbb{E}[f(B_h^z)] - \int_{X_d} \lim_{\varepsilon\to 0} p(u; \varepsilon, z) f(u) \mathrm{d}u \\
&= \mathbb{E}[f(B_h^z)] - f(z)\,.
\end{aligned}$$

Note that we introduced the ε limit to ensure that $p(u; r-s, z)$ was defined for all $t \geq r-s$ (clearly $r = s$ is problematic). The last equality follows from identifying

$$\lim_{\varepsilon\to 0} p(u; \varepsilon, z) = \delta(u - z) \tag{3.9}$$

where $\delta(u)$ is the Dirac delta function. Thus we finally calculate

$$\mathbb{E}[M_{s+h} - M_s|\mathcal{F}_s] = \mathbb{E}[f(B_h^z)] - f(z) - \big(\mathbb{E}[f(B_h^z)] - f(z)\big) = 0\,, \tag{3.10}$$

and so M_s is a martingale, thus concluding the proof.

Part b)

Let $f(x) = \log\|x\|_d = \frac{1}{2}\log\|x\|_d^2$ for $d = 2$ and $g(x) = \|x\|^{2-d}$ for $d \geq 3$. We will show that these are harmonic functions on $\mathbb{R}^d$ respectively, that is $\Delta f = 0 = \Delta g$. We compute

$$\frac{\partial f}{\partial x_1} = \frac{1}{2}\frac{\partial}{\partial x_1}\log(x_1^2 + x_2^2) = \frac{x_1}{x_1^2 + x_2^2}\,, \quad \text{so} \quad \frac{\partial^2 f}{\partial x_1^2} = \frac{x_2^2 - x_1^2}{(x_1^2 + x_2^2)^2} = -\frac{\partial^2 f}{\partial x_2^2} \tag{3.11}$$

where the last equality follows by the symmetry of the form of $\frac{\partial^2}{\partial x_1^2} f$, and so rearranging we have $\Delta f = 0$. For g we may rewrite $g(x) = \|x\|^{-k}$ for some $k \geq 1$, so we have

$$\begin{aligned}
\frac{\partial g}{\partial x_i} &= \frac{\partial}{\partial x_i}\left(\sum_{j=1}^{k+2} x_j^2\right)^{-k/2} = -k x_i \left(\sum_{j=1}^{k+2} x_j^2\right)^{-(k/2+1)} \\
\text{so} \quad \frac{\partial^2 g}{\partial x_i^2} &= \left(\sum_{j=1}^{k+2} x_j^2\right)^{-(k/2+2)} \left(k(k+2)x_i^2 - k\sum_{j=1}^{k+2} x_j^2\right)
\end{aligned}$$

which follows from an unenlightening and lengthy application of the quotient rule. Thus we have

$$\begin{aligned}
\Delta g = \sum_{i=1}^{k+2} \frac{\partial^2 g}{\partial x_i^2} &= \left(\sum_{j=1}^{k+2} x_j^2\right)^{-(k/2+2)} \sum_{i=1}^{k+2} \left(k(k+2)x_i^2 - k\sum_{j=1}^{k} x_j^2\right) \\
&= \left(\sum_{j=1}^{k+2} x_j^2\right)^{-(k/2+2)} \left[k(k+2)\sum_{i=1}^{k+2} x_i^2 - k(k+2)\sum_{i=1}^{k+2} x_i^2\right] = 0\,,
\end{aligned} \tag{3.12}$$

as desired, thus $\Delta g = 0$ and so g is also harmonic.

Part c)

Let $0 < a < |x| < b$ and define $\tau_a^x = \inf\{t > 0 : \|B_t^x\| = a\}$ to be the hitting time of the sphere $S_a \triangleq \{y \in \mathbb{R}^d : |y| = a\}$, and similarly for τ_b^x, by the Brownian motion B^x. We want to calculate $\mathbb{P}(\tau_a^x < \tau_b^x)$ and $\mathbb{P}(\tau_a^x < \infty)$.

For ease let us define our harmonic function generally where

$$f(x) = \begin{cases} \log \|x\| & \text{if } d = 2 \\ \|x\|^{2-d} & \text{if } d \geq 3 \end{cases}, \tag{3.13}$$

which we showed satisfies $\Delta f = 0$ in either case. Then from part a) we know that the process

$$M_t = f(B_t^x) - f(x) \tag{3.14}$$

is a martingale. Let $\tau \triangleq \tau_a^x \wedge \tau_b^x$. We ultimately want to calculate $\mathbb{E}[M_\tau]$, but before we get there we need to consider $M_{\tau\wedge n}$ and then ultimately take the limit as $n \to \infty$ using the dominated convergence theorem.

We begin by noting that in the case $\tau < \infty$, so $\tau \wedge n = \tau$ in the limit, we have

$$|M_\tau| \leq \max\{|f(a) - f(x)|, |f(b) - f(x)|\} \tag{3.15}$$

where we note that $\log a < \log b$ but $a^{-(d-2)} > b^{-(d-2)}$ in their respective cases. Either way, both of these sums are finite and hence we see that M_τ is uniformly bounded. Furthermore in this case we have $\tau \wedge n \to \tau$ in the limit, hence $M_{\tau\wedge n} \to M_\tau$, which is clearly a measurable function.

But by Lemma 2.3, we know that $\sup_{t\geq 0} B_t = +\infty$ almost surely, hence we see that for any fixed a and b we have $\mathbb{P}(\tau < \infty) = 1$ and $\mathbb{P}(\tau = \infty) = 0$, so almost surely we have

$$\lim_{n\to\infty} \mathbb{E}[M_{\tau\wedge n}] = \mathbb{E}[\lim_{n\to\infty} M_{\tau\wedge n}] = \mathbb{E}[M_\tau]\,. \tag{3.16}$$

But then since $\tau \wedge n < \infty$ is a bounded stopping time for any fixed n, we may apply the optional sampling theorem to see that

$$\mathbb{E}[M_\tau] = \mathbb{E}[M_{\tau\wedge n}] = \mathbb{E}[M_{\tau\wedge n}|\mathcal{F}_0] = M_0 = 0\,. \tag{3.17}$$

Calculating the expectation, we have

$$\mathbb{E}[M_\tau] = \mathbb{E}[M_{\tau_a\wedge\tau_b}] = (f(a) - f(x))\mathbb{P}(\tau_a^x < \tau_b^x) + (f(b) - f(x))(1 - \mathbb{P}(\tau_a^x < \tau_b^x)) = 0\,,$$

$$\text{so} \quad \mathbb{P}(\tau_a^x < \tau_b^x) = \frac{f(b) - f(x)}{f(b) - f(a)} = \begin{cases} \dfrac{\log(b) - \log(x)}{\log(b) - \log(a)} & \text{if } d = 2 \\ \dfrac{\|x\|^{-(d-2)} b^{d-2} - 1}{a^{-(d-2)} b^{d-2} - 1} & \text{if } d \geq 3 \end{cases}. \tag{3.18}$$

To calculate $\mathbb{P}(\tau_a^x < \infty)$ we want to take the limit $\tau_b^x \to \infty$, which due to the continuity of Brownian motion is equivalent to $b \to \infty$. We note that for any b we have that as events, $\{\tau_a^x < \tau_b^x\} \subseteq \{\tau_a^x < \tau_{b+\varepsilon}^x\}$ for any $\varepsilon > 0$ and so since this is an expanding sequence of events, by the continuity of probability (so we can bring the limit out in the second equality) we have

$$\mathbb{P}(\tau_a^x < \infty) = \mathbb{P}(\bigcup_{b>0}\{\tau_a^x < \tau_b^x\}) = \lim_{b\to\infty} \mathbb{P}(\tau_a^x < \tau_b^x) = \begin{cases} 1 & \text{if } d = 2 \\ \left(\frac{a}{\|x\|}\right)^{d-2} & \text{if } d \geq 3 \end{cases}. \tag{3.19}$$

In the case of $d \geq 3$ we see that, since $a < \|x\|$ and $d-2 \geq 1$, we have $0 < \mathbb{P}(\tau_a^x < \infty) < 1$, which is stunningly different to $\mathbb{P}(\tau_a^x < \infty) = 1$ in the $d = 2$ case!

Part d)

Let U be a non-empty, bounded open subset of $\mathbb{R}^d$ and define $\sigma_U^x = \sup\{t : B_t^x \in U\}$ to be the last time that B^x visits U. Since open balls form the basis of the topology on $\mathbb{R}^n$, we have that $U = \bigcup_\Lambda \mathcal{B}(x_0, r)$, for some points x_0, radii r, and a countable set Λ, where $\mathcal{B}(x_0, r)$ is the ball of radius r centred at x_0. Thus to analyse $\mathbb{P}(\sigma_U^x < \infty)$ and $\mathbb{P}(\sigma_U^x = \infty)$ it suffices to consider a single open ball $U = \mathcal{B}(x_0, r)$ for some given x_0 and r. We know that a shifted Brownian motion is another Brownian motion, so we have for any $r > 0$

$$\tau_r^x = \inf\{t > 0 : B_t^x \in B(0, r)\} = \inf\{t > 0 : B_t^{x-x_0} \in B(x_0, r)\}\,. \tag{3.20}$$

Consider the $d = 2$ case first. Then we know that τ_a^x is finite almost surely. Let $t_1 = \tau_a^x$ be the stopping time for the first time this occurs. But then Brownian motion is time invariant by the strong markov property, that is, $B^{x,(1)} = \{B_{t+1} - B_1 : t \geq 0\}$ is a Brownian motion independent of $\mathcal{F}_1$. Therefore, we see that $t_2 = \inf\{t > 0 : B_t^{x-x_0,(1)} \in \mathcal{B}(x_0, r)\}$ is again finite almost surely. Continuing in this way, we have a sequence of times $t_n = \inf\{t > n - 1 : B^{x-x_0} \in \mathcal{B}(x_0, r)\}$ that almost surely converges to ∞ - in other words, B^x almost surely returns to $\mathcal{B}(x_0, a)$ infinitely often, meaning the last time it visits U is at infinity, hence in $d = 2$ we have $\mathbb{P}(\sigma_U^x = \infty) = 1$.

Recall that in the $d \geq 3$ case we have $0 < \mathbb{P}(\tau_a^x < \infty) < 1$. As explained above, we may without loss of generality consider the hitting time of $\mathcal{B}(0, r)$. We may consider events of the form

$$A_n = \{\|B_t^x\| > n \text{ for all } t \geq \tau_{n^3}^x\}\,. \tag{3.21}$$

Again by Lemma 2.3 we know that $\tau_{n^3}^x$ is finite almost surely. Then for any $n^3 \geq \|x\|$ (so B starts inside S_{n^3}) we have

$$\begin{aligned}
\mathbb{P}(A_n^c) &= \mathbb{E}[\mathbb{1}(\exists\, t \geq \tau_{n^3}^x \text{ s.t. } \|B_t^x\| = n)] \\
&= \mathbb{E}[\mathbb{E}[\mathbb{1}(\exists\, t \geq \tau_{n^3}^x \text{ s.t. } \|B_t^x\| = n)|B_{\tau_{n^3}^x} = z]] \\
&= \mathbb{E}[\mathbb{P}(\exists\, t \geq \tau_{n^3}^x \text{ s.t. } \|B_t^x\| = n|B_{\tau_{n^3}^x} = z)] \\
&= \mathbb{E}[\mathbb{P}(\exists\, t \geq 0 \text{ s.t. } \|B_t^z\| = n)] \\
&= \mathbb{E}[\mathbb{P}(\tau_n^z < \infty)] = \mathbb{E}\left[\left(\frac{n}{n^3}\right)^{d-2}\right] = \left(\frac{1}{n^2}\right)^{d-2}\,,
\end{aligned} \tag{3.22}$$

where we used the tower property in the second equality, the strong markov property of Brownian motion in the fourth equality, and our result from part c) for $d \geq 3$ in the sixth equality since $\|z\| = n^3$ by conditional assumption. But then since $\mathbb{P}(A_n^c) = \left(\frac{1}{n^2}\right)^{d-2}$ is summable, that is, $\sum_{n=1}^\infty \left(\frac{1}{n^2}\right)^{d-2} < \infty$, the Borel-Cantelli lemma tells us that only finitely many A_n^c happen almost surely. In other words, for only finitely many n does B return, which means that we must have that $\|B_t\| \to \infty$ almost surely. Thus we see that $\mathbb{P}(\sigma_U^x < \infty) = 1$ for any arbitrary bounded open set $U \subseteq \mathbb{R}^d$.

Part e)

Let $y \in \mathbb{R}^d$ and define $\zeta_y^x \triangleq \inf\{t > 0 : B_t^x = y\}$, we want to show that $\mathbb{P}(\zeta_y^x < \infty) = 0$ for all $d \geq 2$. To do this we can consider $\mathcal{B}(y, a)$ and take the limit $a \to 0$. Using our shifting argument from the previous part, we need only consider the case $y = 0$, so let $U = \mathcal{B}(0, a)$ and consider $\zeta_0^x = \lim_{a \to 0} \tau_a^x$. To start with, let $x \neq 0$ and let $b > 0$ be fixed, then we have

$$\mathbb{P}(\zeta_0^x < \tau_b^x) \leq \mathbb{P}(\lim_{a \to 0} \tau_a^x < \tau_b^x) = \lim_{a \to 0} \mathbb{P}(\tau_a^x < \tau_b^x) = 0 \tag{3.23}$$

where the last equality holds when observing the limits in (3.18), in both cases (ultimately $f(a)$, on the denominator in both expressions, diverges as $a \to 0$ in both cases). Taking the limit $b \to \infty$ of both sides we see that $\mathbb{P}(\zeta_0^x < \infty) = 0$ for all $d \geq 2$.

In the case where $x = 0$, we can use an identical Markov property argument as in part d). Let $\varepsilon > 0$, then

$$\begin{aligned} \mathbb{P}(B_t^0 = 0 \text{ for some } t \geq \varepsilon) &= \mathbb{E}[\mathbb{E}[\mathbb{1}(B_t^0 = 0 \text{ for some } t \geq \varepsilon) | B_\varepsilon = x]] \\ &= \mathbb{E}[\mathbb{P}(B_t^x = 0 \text{ for some } t \geq 0)] = 0\,. \end{aligned} \tag{3.24}$$

Therefore we can finally take the limit

$$\mathbb{P}(\zeta_0^0 < \infty) = \lim_{\varepsilon \to 0} \mathbb{P}(B_t^0 = 0 \text{ for some } t \geq \varepsilon) = 0 \tag{3.25}$$

and so for all $d \geq 2$ we have $\mathbb{P}(\zeta_y^x < \infty) = 0$. $\square$

4. Question 25 - Stochastic integral decomposition (W2)

Let $B = \{B_t : 0 \leq t \leq 1\}$ be a one dimensional Brownian motion.

Part a)

Let $Y \triangleq \int_0^1 B_t dt$. We want to find the unique progressively measurable process Φ such that

$$Y = \mathbb{E}[Y] + \int_0^1 \Phi_t dB_t \,, \tag{4.1}$$

which is guaranteed to exist and be unique due to Proposition 3.7. Noting that by Fubini's theorem we have

$$\mathbb{E}\left[\int_0^1 B_t dt\right] = \int_0^1 \mathbb{E}[B_t] dt = 0 \,,$$

this shows that we want the unique Φ_t such that

$$\int_0^1 \Phi_t dB_t = \int_0^1 B_t dt \,. \tag{4.2}$$

Theorem 3.2 (Itô's formula) states that for a $C^{1,2}$ function $f : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ we have

$$f(t, B_t) - f(s, B_s) = \int_s^t \partial_t f(u, B_u) du + \int_s^t \partial x f(u, B_u) dB_u + \frac{1}{2} \int_s^t \partial_x^2 f(u, B_u) dt \,, \tag{4.3}$$

and so comparing terms shows that we want to find $f(t, x)$ such that $\partial_t f = x$, $\partial_x f = -\Phi_t$ and $\partial_x^2 f = 0$. We claim that the function

$$f(t, x) = tx \,, \quad \text{so} \quad f(t, B_t) = tB_t \tag{4.4}$$

satisfies our desired equalities. Applying (4.3) to f, noting that $B_0 = 0$ and $\partial_x^2 f = 0$, we have

$$f(1, B_1) - f(0, B_0) = B_1 = \int_0^1 B_t dt + \int_0^1 t \, dB_t \,. \tag{4.5}$$

We know by definition that

$$\int_0^1 dB_t = B_1 - B_0 = B_1 \,, \tag{4.6}$$

hence we may express (4.5) as

$$\int_0^1 B_t dt = B_1 - \int_0^1 t \, dB_t = \int_0^1 (1 - t) \, dB_t \,, \tag{4.7}$$

and so taking $\Phi_t = 1 - t$ we have found our unique process that satisfies (4.1). It is clear that Φ_t is $\{\mathcal{F}_t\}$-measurable, and that it has continuous sample paths and so we are done.

Part b)

Define $S_a \triangleq \sup_{0\le t\le a} B_t$. We first want write $\mathbb{E}[S_1|\mathcal{F}_t^B]$ as a function of (t, S_t, B_t).

We see that for any $t \le 1$ we can break the supremum up into

$$S_1 = S_t + \left(\sup_{t\le u\le 1}(B_u - S_t)\right)^+ , \tag{4.8}$$

where $^+$ denotes the positive part, i.e. $x^+ = \max(0, x)$, which is easy to convince oneself of with enough staring and diagrams. We can then write

$$\begin{aligned}
\mathbb{E}[S_1|\mathcal{F}_t^B] &= \mathbb{E}\left[S_t + \left(\sup_{t\le u\le 1}(B_u - S_t)\right)^+ \middle| \mathcal{F}_t^B\right] \\
&= S_t + \mathbb{E}\left[\left(\sup_{t\le u\le 1}\big((B_u - B_t) - (S_t - B_t)\big)\right)^+ \middle| \mathcal{F}_t^b, S_t = s, , B_t = b\right] \\
&= S_t + \mathbb{E}\left[\left(\sup_{t\le u\le 1}(B_u - B_t) - (s - b)\right)^+\right]
\end{aligned}$$

where we used the $\mathcal{F}_t^B$ measurability of S_t and B_t throughout, and in the third line we used the fact that $B_u - B_t$ is independent of $\mathcal{F}_t^B$ and properties of sup to bring the constants out. Noting that $t \le u \le 1$ is the same as $0 \le u - t \le 1 - t$, and $B_u - B_t \sim B_{u-t}$ is another Brownian motion by the strong Markov property, we have

$$\mathbb{E}[S_1|\mathcal{F}_t^B] = S_t + \mathbb{E}[(S_{1-t} - (s - b))^+] . \tag{4.9}$$

To calculate this expectation we note the tail probability formula for any integrable random variable X, $\mathbb{E}[X^+] = \int_0^\infty \mathbb{P}(X > x)dx$, so with a simple change of variables we have for any constant c that $\mathbb{E}[(X-c)^+] = \int_0^\infty \mathbb{P}(X-c > x)dx = \int_c^\infty \mathbb{P}(X > x)dx = \int_c^\infty (1-F_X(x))dx$. So we finally arrive at

$$\mathbb{E}[S_1|\mathcal{F}_t^B] = S_t + \int_{S_t - B_t}^\infty (1 - F_{S_{1-t}}(x))dx \triangleq f(t, S_t, B_t) , \tag{4.10}$$

which we note by Exercise 6.3 is a well defined martingale since S_1 is an integrable random variable. Since we desire a unique progressively measurable process Φ such that $S_1 = \mathbb{E}[S_1] + \int_0^1 \Phi_t dB_t$, we clearly want to now apply Itô's formula to our above $f(t, S_t, B_t)$.

We can then appeal to the Doob-Meyer decomposition theorem for martingales (usually stated for submartingales), which states that if we have martingales M_t and N_t, and a stochastic process A_t, if $M_t = N_t + A_t$ then we must have $A_t \equiv 0$. When we look at Itô's formula, we know that the dB_t term (i.e. the stochastic integral part) is a martingale. Therefore, since $f(t, S_t, B_t)$ is a martingale, we see that when we apply Itô's formula we *only need to consider* the dB_t term due to the Doob-Meyer theorem above. Therefore we have

$$\mathrm{d}f(t, S_t, B_t) = \frac{\partial}{\partial B_t} f(t, S_t, B_t) dB_t . \tag{4.11}$$

To calculate the partial derivative we can make a change of variables $u = -x + S_t$, so $dx = -du$, which using the fundaemental theorem of calculus gives

$$\frac{\partial}{\partial B_t} f(t, S_t, B_t) = \frac{\partial}{\partial B_t} \int_{B_t}^{-\infty} -(1 - F_{S_{1-t}}(-u + S_t))du = 1 - F_{S_{1-t}}(S_t - B_t) . \tag{4.12}$$

Integrating both sides of (4.11) from 0 to t we have

$$\int_0^t \mathrm{d}f(s, S_s, B_s) = \int_0^t \frac{\partial}{\partial B_s} f(s, S_s, B_s) dB_s ,$$

$$\text{so} \quad f(t, S_t, B_t) - f(0, S_0, B_0) = \int_0^t (1 - F_{S_{1-s}}(S_s - B_s))dB_s ,$$

but then noting that we have

$$f(0, S_0, B_0) = \mathbb{E}[S_1 | \mathcal{F}_0^B] = \mathbb{E}[S_1] \tag{4.13}$$

since we are conditioning on the trivial sigma algebra, we may finally write

$$\mathbb{E}[S_1 | \mathcal{F}_t^B] = \mathbb{E}[S_1] + \int_0^t (1 - F_{S_{1-s}}(S_s - B_s))dB_s . \tag{4.14}$$

We are nearly done, but to simplify this a bit further we may evaluate the inside of the integrand. Recall from lectures that for $x \geq 0$ (since $S_t \geq 0$ for any t since B starts at 0), we have the identity $\mathbb{P}(S_t \geq x) = 2\mathbb{P}(B_t \geq x)$, so

$$F_{S_t}(u) = \int_0^u \frac{2}{\sqrt{2\pi t}} e^{-\frac{y^2}{2t}} dy ,$$

meaning we can write in terms of the standard normal CDF $\Phi^Z(x) := \mathbb{P}(Z \leq x)$,

$$F_{S_{1-s}}(S_s - B_s) = \int_0^{S_s - B_s} \frac{2}{\sqrt{2\pi(1-s)}} e^{-\frac{y^2}{2(1-s)}} dy = 2\left(\Phi^Z\left(\frac{S_S - B_S}{\sqrt{1-s}}\right) - \frac{1}{2}\right) , \tag{4.15}$$

where the second equality is because we have a Gaussian integral with standard deviation $\sqrt{1-s}$ (but only on $[0, S_s - B_s]$ instead of $[-\infty, S_s - B_s]$). Therefore we finally see that for all $t \leq 1$ we have

$$\mathbb{E}[S_1 | \mathcal{F}_t^B] = \mathbb{E}[S_1] + 2\int_0^t \left(1 - \Phi^Z\left(\frac{S_S - B_S}{\sqrt{1-s}}\right)\right) dB_s ,$$

but in particular we can take $t = 1$, meaning $\mathbb{E}[S_1 | \mathcal{F}_1^B] = S_1$ (since S_1 is $\mathcal{F}_1^B$ measurable), hence we have

$$S_1 = \mathbb{E}[S_1] + \int_0^1 \underbrace{2\left(1 - \Phi^Z\left(\frac{S_S - B_S}{\sqrt{1-s}}\right)\right)}_{\Phi_t} dB_s . \tag{4.16}$$

so our unique progressively measurable random process is precisely the function in the integrand (not to confuse notation with the normal CDF Φ^Z) and so we are done. □

5. Question 35 - Stochastic harmonic oscillator (W1)

Part a)

Let $A \in M_{n\times n}(\mathbb{R})$ be a deterministic matrix that does not depend on time, $a(t) \in M_{n\times 1}(\mathbb{R})$ and $\sigma(t) \in M_{n\times d}(\mathbb{R})$ be bounded, deterministic functions of time and let B be a d-dimensional Brownian motion. We will consider the linear SDE

$$dX_t = \big(AX_t + a(t)\big)dt + \sigma(t)dB_t\,. \tag{5.1}$$

We start by obtaining the stochastic integrating factor given by

$$Z_t \triangleq \exp\left(\int_0^t A ds\right) = \exp\left(At\right)\,, \quad \text{so} \quad Z_t^{-1} = \exp(-At)\,. \tag{5.2}$$

Using Itô's formula and the product rule, noting that $d(\exp(At)) = A\exp(At)dt = \exp(At)Adt$, we have

$$\begin{aligned} d(Z_t^{-1}X_t) &= d(Z_t^{-1})X_t + Z_t^{-1}dX_t + d(Z_t^{-1})dX_t \\ &= -e^{-At}AX_tdt + \Big(e^{-At} - e^{-At}Adt\Big)\Big(\big(AX_t + a(t)\big)dt + \sigma(t)dB_t\Big) \\ &= e^{-At}a(t)dt + e^{-At}\sigma(t)dB_t\,, \end{aligned} \tag{5.3}$$

where the third equality follows from the fact that $(dt)^2$ and $(dt)(dB_t) \approx (dt)^{3/2}$ are negligilbe compared to the higher order terms. Thus integrating both sides from 0 to t we have

$$\begin{aligned} e^{-At}X_t - X_0 &= \int_0^t e^{-As}a(s)ds + \int_0^t e^{-As}\sigma(s)dB_s\,, \\ \text{so} \quad X_t &= e^{At}X_0 + \int_0^t e^{A(t-s)}a(s)ds + \int_0^t e^{A(t-s)}\sigma(s)dB_s\,. \end{aligned} \tag{5.4}$$

Part b)

Consider the equation for the stochastic harmonic oscillator where B_t is a one-dimensional Brownian motion,

$$\begin{cases} dX_t = Y_tdt\,, \\ mdY_t = -kX_tdt - cY_tdt + \sigma dB_t \end{cases}\,, \tag{5.5}$$

where $m, k, c, \sigma > 0$ are positive real constants. In light of part a), we can reformulate this into a matrix equation given by

$$d\begin{pmatrix} X_t \\ Y_t \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{pmatrix}\begin{pmatrix} X_t \\ Y_t \end{pmatrix}dt + \begin{pmatrix} 0 \\ \frac{\sigma}{m} \end{pmatrix}dB_t\,. \tag{5.6}$$

Identifying this with (5.1) we may set

$$X_t = \begin{pmatrix} X_t \\ Y_t \end{pmatrix}\,, \quad A = \begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{pmatrix}\,, \quad a(t) = 0\,, \quad \sigma(t) = \begin{pmatrix} 0 \\ \frac{\sigma}{m} \end{pmatrix}\,. \tag{5.7}$$

Hence using our derived formula in (5.4), we have

$$X_t = e^{At}\left(\begin{pmatrix} X_0 \\ Y_0 \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{\sigma}{m} \end{pmatrix}\int_0^t e^{-s}dB_s\right)\,. \tag{5.8}$$

It is perfectly fine to leave the equation in this form, but for the sake of curiosity I will present the final form of e^{At} without presenting many interluding calculations. To calculate the matrix exponential we first want to diagonalise the matrix such that $A = P^{-1}DP$ where P is the change of basis matrix with eigenvectors in the columns and D is the diagonal matrix with eigenvalues on the diagonal (supposing that A is non-degenerate, so $k \neq 0$). We see that the eigenvalues of A are

$$\lambda_{1,2} = -\frac{c}{2m} \pm \frac{\sqrt{c^2 - 4mk}}{2m} =: -s \pm \omega \,. \tag{5.9}$$

With a straightforward calculation this gives eigenvectors of

$$v_1 = \begin{pmatrix} 1 \\ \lambda_1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 1 \\ \lambda_2 \end{pmatrix}, \quad \text{so} \quad P = \begin{pmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{pmatrix}. \tag{5.10}$$

Hence we can calculate

$$\begin{aligned} \exp(At) &= \exp(P^{-1}DPt) = P^{-1}\exp(Dt)P \\ &= e^{-st}\begin{pmatrix} \cosh(\omega t) + \frac{s}{\omega}\sinh(\omega t) & -\left(\frac{s}{\omega}+1\right)\sinh(\omega t) \\ \left(1-\frac{s}{\omega}\right)\sinh(\omega t) & \cosh(\omega t) - \frac{s}{\omega}\sinh(\omega t) \end{pmatrix}. \end{aligned} \tag{5.11}$$

Since we know that $P_t := e^{At}$ is the "transition" matrix associated to the SDE, we must have that each entry satisfies $0 \leq (P_t)_{i,j} \leq 1$, which can be checked with a routine calculation involving the hyperbolic trigonometric identities - we leave this as an exercise for the reader.

Furthermore, we can elaborate on the integral part of (5.8). By Itô's formula we have that

$$d(e^{-s}B_s) = -e^{-s}B_s ds + e^{-s}dB_s \,, \quad \text{so} \quad \int_0^t e^{-s}dB_s = e^{-t}B_t + \int_0^t e^{-s}B_s ds \,. \tag{5.12}$$

We can then use (5.11) and (5.12) to find the more explicit form of X_t, which doesn't fit in the margin so we will leave it in the more simplified form of (5.8)! □