Partial Differential Equations Assignment 2

Liam Carroll - 830916

Due Date: 17$^{\text{th}}$ September 2021

Q1. Finite speed of propagation

Let $B(x_0, r_0)$ be the closed ball of radius r_0 centred at $x_0 \in \mathbb{R}^d$. Consider the domain of dependence,

$$\mathcal{D}(B(x_0, r_0)) = \{(t, x) : 0 \le t \le r_0\,,\ |x - x_0| \le r_0 - t\}\ .$$

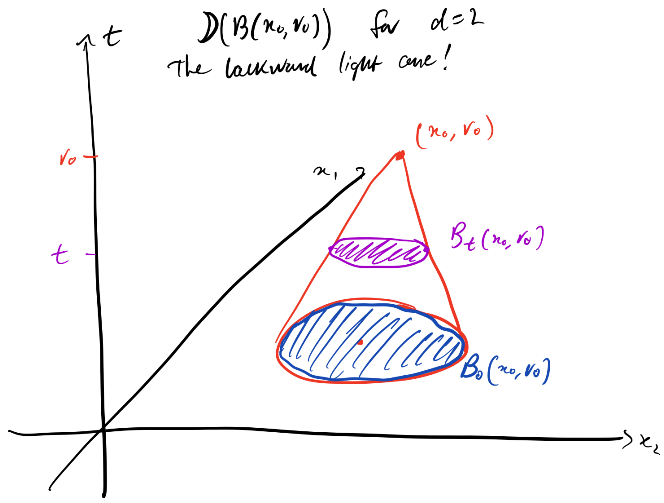

Figure 1: Note that $\mathcal{D}(B(x_0, r_0))$ is inclusive of everything inside the red cone (it is closed).

We will show that if $u(t, x)$ is a solution of the wave equation with zero boundary conditions,

$$\frac{\partial^2 u}{\partial t^2} = \Delta u\,, \quad u(0, x) = \partial_t u(0, x) = 0\,, \quad \text{for all } x \in B(x_0, r_0)\,,$$

then $u(t, x) = 0$ for all $(t, x) \in \mathcal{D}(B(x_0, r_0))$, where $x \in \mathbb{R}^d$.

Part a)

Let

$$B_t(x_0, r_0) = \{x : |x - x_0| \le r_0 - t\}$$

be $\mathcal{D}(B(x_0, r_0))$ for a fixed value t. For sanity, note that $\{x_0\} = B_{r_0} \subset B_0(x_0, r_0)$. Consider the energy integral

$$E(t) = \int_{B_t(x_0, r_0)} e(t, x)dx\,, \qquad \text{where} \quad e(t, x) = \frac{1}{2}\left((\partial_t u)^2 + |\nabla u|^2\right)(t, x)\,.$$

Firstly, recall the coarea formula on the sphere (Stein, Shakarchi Fourier Analysis pg293) which states that for any continuous and integrable function $f(x)$ we may write for any ball $\mathcal{B}(x_0, R)$ of radius R centred at x_0

$$\int_{\mathcal{B}(x_0,R)} f(x)\,\mathrm{d}x = \int_0^R \left(\int_{\partial\mathcal{B}(x_0,R)} f(x)\,\mathrm{d}\sigma \right) \mathrm{d}r\,.$$

In essence, this line is where the geometry of the light cone is exploited. It just so happens that the sphere obeys this relation - most (maybe all?) other domains of integration would give a Jacobian that does not cancel out with the spherical measure $\mathrm{d}\sigma(\gamma)$. Let us write $B_t(x_0, r_0) = \mathcal{B}(x_0, r_0 - t)$ so as to not confuse ourselves. Then using the coarea formula,

$$E(t) = \int_0^{r_0-t} \left(\int_{\partial\mathcal{B}(x_0,r)} e(t,x)\,\mathrm{d}\sigma \right) \mathrm{d}r\,.$$

Let $f(x, r)$ denote the inner integral in the above equation. We may then appeal to the Leibniz integral rule (or as Volker puts it, merely the fundamental theorem of calculus) to write

$$\begin{aligned}
E'(t) &= \int_0^{r_0-t} \frac{\partial}{\partial t} f(x,r)\,\mathrm{d}r + f(x, r_0 - t)\frac{\partial}{\partial t}(r_0 - t) \\
&= \int_0^{r_0-t} \left(\frac{\partial}{\partial t} \int_{\partial\mathcal{B}(x_0,r)} e(t,x)\,\mathrm{d}\sigma \right) \mathrm{d}r - \int_{\partial\mathcal{B}(x_0,r_0-t)} e(t,x)\,\mathrm{d}\sigma \\
&= \int_0^{r_0-t} \left(\int_{\partial\mathcal{B}(x_0,r)} \frac{\partial}{\partial t} e(t,x)\,\mathrm{d}\sigma \right) \mathrm{d}r - \int_{\partial\mathcal{B}(x_0,r_0-t)} e(t,x)\,\mathrm{d}\sigma \\
&= \int_{B_t(x_0,r_0)} \partial_t e(t,x)\,\mathrm{d}x - \int_{\partial B_t(x_0,r_0)} e(t,x)\,\mathrm{d}\sigma\,,
\end{aligned}$$

where we may take the ∂_t inside the integral since $e(t, x)$ is assumed to be continuous and integrable, and the bounds are independent of t.

Part b)

A simple calculation using the chain rule shows

$$\partial_t e(t,x) = \left(\frac{\partial u}{\partial t}\right)\left(\frac{\partial^2 u}{\partial t^2}\right) + \sum_{i=1}^{d} \left(\frac{\partial}{\partial x_i}\frac{\partial u}{\partial t}\right)\left(\frac{\partial u}{\partial x_i}\right) = (\partial_t u)(\Delta u) + (\nabla \partial_t u)\cdot(\nabla u)\,,$$

where the last equality follows from the fact that u satisfies the wave equation. Recall the identity $\nabla \cdot (f\mathbf{F}) = f\nabla \cdot \mathbf{F} + (\nabla f) \cdot \mathbf{F}$, which we can use to write

$$\partial_t e(t,x) = \nabla \cdot (\partial_t u \nabla u)\,.$$

Thus, using the divergence theorem we have

$$\int_{B_t(x_0,r_0)} \partial_t e(t,x)\,\mathrm{d}x = \int_{B_t(x_0,r_0)} \nabla \cdot (\partial_t u \nabla u)\,\mathrm{d}x = \int_{\partial B_t(x_0,r_0)} (\partial_t u \nabla u)\cdot n\,\mathrm{d}\sigma\,,$$

where n is the unit normal vector to the boundary.

Part c)

We have now shown that

$$E'(t) = \int_{\partial B_t(x_0,r_0)} \big((\partial_t u \nabla u)\cdot n - e(t,x)\big)\, \mathrm{d}\sigma\,.$$

Using the Cauchy-Schwarz inequality, $a\cdot b \le |a||b|$, we have

$$\begin{aligned}(\partial_t u\nabla u)\cdot n - e(t,x) &\le |\partial_t u\nabla u||n| - e(t,x)\\ &= |\partial_t u||\nabla u| - \frac{1}{2}|\partial_t u|^2 - \frac{1}{2}|\nabla u|^2\\ &= -\frac{1}{2}(|\partial_t u| - |\nabla u|)^2 \le 0\,.\end{aligned}$$

Since $E'(t)$ is an integral over a non-positive function, we thus have $E'(t)\le 0$.

To finally see uniqueness on $\mathcal{D}(B(x_0,r_0))$, note that since $u(0,x)=0$ everywhere on $B(x_0,r_0)$ we also must have $\partial_{x_i}u(0,x)=0$. Combining this with $\partial_t u(0,x)=0$ we thus have $e(0,x)=0$ and so $E(0)=0$. But since $e(t,x)\ge 0$, we must have $E(t)\ge 0$, but since we have shown that $E'(t)\le 0$ for all t we necessarily have $E(t)=0$ for all t. This implies $e(t,x)=0$ everywhere, thus all partials must be 0, implying $u(t,x)$ is necessarily a constant. But since $u(0,x)=0$, we finally conclude that $u(t,x)=0$ everywhere as required.

Q2. Weak convergence in Hilbert space

Let $\mathcal{H}$ be an infinite-dimensional Hilbert space, for which we know that the unit ball is not compact in $\mathcal{H}$. However, we can show a kind of weak compactness. Let $\{f_n\}$ be a sequence in $\mathcal{H}$ on the unit ball, so $\|f_n\|=1$ for all n. We will show that there exists an $f\in\mathcal{H}$ and a subsequence $\{f_{n_k}\}$ such that for all $g\in\mathcal{H}$ we have weak convergence, $\lim_{k\to\infty}(f_{n_k},g)=(f,g)$.

Let $\{e_j\}$ be an orthonormal basis for $\mathcal{H}$, where $e_j=(0,\dots,0,1,0,\dots,0)$ such that the 1 occurs in the j entry as usual. Define a sequence by $x_n^1=\{(f_n,e_1)\}_{n=1}^\infty\subseteq\mathbb{C}$. Since f_n is on the unit ball for each n, we have by Cauchy-Schwarz that $|(f_n,e_1)|\le\|f_n\|\|e_1\|=1$. This implies that x_n^1 is a sequence in the compact ball $B(0,1)\subseteq\mathbb{C}$, and thus has a convergent subsequence which we may denote $x_{n_j}^1=\{(f_{n,j},e_1)\}_{j=1}^\infty$, where $x_{n_j}^1\to X^1$ for some $X^1\in B(0,1)\subseteq\mathbb{C}$.

Then consider the sequence $x_{n_j}^2=\{(f_{n_j},e_2)\}_{j=1}^\infty$. By the same argument, this also has a convergent subsequence $x_{n_{j_i}}^2\to X^2$ as a limit in i. Moreover, $\{f_{n_{j_i}}\}_{i=1}^\infty$ is a subsequence of $\{f_{n_j}\}_{j=1}^\infty$ in $\mathcal{H}$, and since subsequences must converge to the same element as the sequence, we have $x_{n_{j_i}}^1\to X^1$.

Repeating this process indefinitely, we can construct an infinite matrix of rows $\{f_{k,i}\}_{i=1}^\infty$ such that:

- f_k is a subsequence of each previous row $f_1,\dots,f_{k-1}$,
- $\lim_{i\to\infty}(f_{k,i},e_k)=X^k$,
- and $\lim_{i\to\infty}(f_{k+m,i},e_k)=X^k$ for all $m\ge 0$

If we then define the diagonal subsequence $f_{n,n}$ of f_n, then for any fixed e_k we have $\lim_{n\to\infty}(f_{n,n}, e_k) = X^k$ by construction. In other words, for any basis element e_k, the subsequence $\{(f_{n,n}, e_k)\}_{n=1}^{\infty}$ converges!

Let $g \in \mathcal{H}$ be an arbitrary element, which we may write as $g = \sum_{k=1}^{\infty} g_k e_k$ where $g_k = (g, e_k)$. Let $S_K(g) = \sum_{k=1}^{K} g_k e_k$. We can show that the sequence (f_{nn}, g) is Cauchy. Let $\varepsilon > 0$ be fixed. For any $n > m$ we can estimate

$$\begin{aligned}|(f_{nn}, g) - (f_{mm}, g)| &\leq |(f_{nn}, g) - (f_{nn}, S_K(g))| + |(f_{nn}, S_K(g)) - (f_{mm}, S_K(g))| \\ &\quad + |(f_{mm}, g) - (f_{mm}, S_K(g))| \\ &= |(f_{nn}, g - S_K(g))| + |(f_{nn}, S_K(g)) - (f_{mm}, S_K(g))| + |(f_{mm}, g - S_K(g))| \\ &\leq |f_{nn}||g - S_K(g)| + |(f_{nn}, S_K(g)) - (f_{mm}, S_K(g))| + |f_{mm}||g - S_K(g)|\end{aligned}$$

Since $S_K(g) \to g$, there is some K such that the first and last terms are less than $\varepsilon/2$. For the second term, taking $N = K$ we have

$$|(f_{nn}, S_K(g)) - (f_{mm}, S_K(g))| = \left|\sum_{k=1}^{K} g_k[(f_{nn}, e_k) - (f_{mm}, e_k)]\right| = \left|\sum_{k=1}^{K} g_k[X^k - X^k]\right| = 0\,.$$

Thus for all $n, m \geq K$ we have $|(f_{nn}, g) - (f_{mm}, g)| < \varepsilon$, showing that (f_{nn}, g) is Cauchy. Finally, by the Riesz representation theorem, we may define

$$\ell_n : \mathcal{H} \to \mathbb{C}\,, \quad \ell_n(g) = (f_{n,n}, g)\,.$$

Then since the dual $\mathcal{H}^*$ is also a Hilbert space, and ℓ_n is Cauchy, we have that

$$\lim_{n\to\infty} = \ell_n = \ell \in \mathcal{H}^*$$

for some ℓ. By reversing Riesz we thus have some $f \in \mathcal{H}$ such that $\ell(g) = (f, g)$ for all $g \in \mathcal{H}$. This shows that f_n converges weakly to f. $\square$