Lie Algebras Assignment 2

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Lecture 3

Q1. Everything rotates

Part a)

Consider the map

$$\begin{aligned} \Psi : S^2 \times [0, 2\pi) &\to \mathrm{SO}(3) \\ (\hat{n}, \alpha) &\mapsto R^{\hat{n}}_{\alpha} \end{aligned} \tag{1.1}$$

where for a unit vector $\hat{n} = R^z_{\psi} R^y_{\theta - \frac{\pi}{2}}(e_1)$ (for $0 \leq \theta \leq \pi$ and $0 \leq \psi < 2\pi$) and angle $\alpha \in [0, 2\pi)$ we define

$$R^{\hat{n}}_{\alpha} = R^z_{\psi} R^y_{\theta - \frac{\pi}{2}} R^x_{\alpha} R^y_{\frac{\pi}{2} - \theta} R^z_{-\psi}\,. \tag{1.2}$$

Note that here $R^{e_i}_{\beta}$ is the rotation matrix about the axis e_i (i.e. x, y, z) of angle β, as defined in (L3, p8). Our task is to show that Ψ is surjective - that is, given an arbitrary element $A \in \mathrm{SO}(3)$, we want to show that $A = R^{\hat{n}}_{\alpha}$ for some α and $\hat{n}$.

Let $A \in \mathrm{SO}(3)$, then $A : \mathbb{R}^3 \to \mathbb{R}^3$ satisfies $\langle Ax, Ay \rangle = \langle x, y \rangle$ for all vectors $x, y \in \mathbb{R}^3$ (where $\langle x, y \rangle = \sum_{i=1}^3 x_i y_i$), and also $\det(A) = \lambda_1 \lambda_2 \lambda_3 = 1$ where λ_i are the eigenvalues of A. We first show that at least one eigenvalue satisfies $\lambda_1 = 1$ (without loss of generality). Suppose v is an eigenvector of A, then $Av = \lambda v$ for some eigenvalue $\lambda \in \mathbb{C}$ so

$$\langle v, v \rangle = \langle Av, Av \rangle = \langle \lambda v, \lambda v \rangle = \lambda \overline{\lambda} \langle v, v \rangle\,, \;\text{ so }\; |\lambda| = 1 \;\text{ for all eigenvalues}\,. \tag{1.3}$$

Since A is a real-valued matrix, its characteristic polynomial $P(X) = \sum_{j=0}^3 a_j X^j$ will be a cubic with real coefficients $a_j \in \mathbb{R}$, hence has at least one real root. As such, at least one eigenvalue is real, say $\lambda_1 \in \mathbb{R}$, and so by (1.3) we have $\lambda_1 = \pm 1$. Since $\det(A) = \lambda_1 \lambda_2 \lambda_3 = 1$ we are then in two cases.

If $\lambda_1, \lambda_2, \lambda_3$ are all real, then each $\lambda_i \in \{-1, 1\}$. But if $\lambda_i = -1$ for all i then $\lambda_1 \lambda_2 \lambda_3 = -1$, hence a contradiction, so at least one λ_i must be 1, say $\lambda_1 = 1$. If λ_2 and λ_3 are both non-real, since $\lambda_1 \lambda_2 \lambda_3 = e^{i(\theta_1 + \theta_2 + \theta_3)} = e^{0i} = 1$ we must have $\theta_1 + \theta_2 + \theta_3 = 0$. But by the complex conjugate root theorem, since $P(X)$ has real coefficients the roots must come in complex conjugate pairs, so $\theta_2 = -\theta_3$ and so $\theta_1 = 0$, hence $\lambda_1 = 1$. Therefore, at least one eigenvalue satisfies $\lambda_1 = 1$.

We may then consider how A acts on an arbitrary vector $n \in \mathbb{R}^3$, which we may restrict to unit vectors $\hat{n}$ since we are working on the sphere. By the above, there is some unit vector such that $A\hat{n} = \hat{n}$. For the moment, suppose that we happened to have $\hat{n} = e_1$. Then since $Ae_1 = e_1$, the first column of A must be e_1 itself. But then since A is orthogonal we have $A^{-1} = A^T$ and so $e_1 = A^{-1}Ae_1 = A^{-1}e_1 = A^Te_1$, and so the first column of A^T must also be e_i, which is to say the first row of A is e_1 too. In other words, we necessarily have

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{pmatrix} . \tag{1.4}$$

Let $B : \mathbb{R}^2 \to \mathbb{R}^2$ denote the bottom right block matrix in (1.4). Then we have $1 = \det(A) = \det(B)$ by the usual cofactor expansion, and since $A^TA = \mathbb{1}_3$, we see by basic operations with block matrices that we must also have $B^TB = \mathbb{1}_2$, hence $B \in \mathrm{SO}(2)$. As such, B must have the standard form of a 2×2 rotation matrix since both rows must be orthogonal unit vectors that are norm preserving, hence we have

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{pmatrix} = R^x_\alpha \quad \text{for some } \ \alpha \in [0, 2\pi) . \tag{1.5}$$

Now instead suppose that $\hat{n} = R^z_\psi R^y_{\theta - \frac{\pi}{2}}(e_1)$ is an arbitrary unit eigenvector of A for some ψ, θ. Then we can write $e_1 = R^y_{\frac{\pi}{2} - \theta} R^z_{-\psi}\hat{n}$, and so since $A\hat{n} = \hat{n}$ we have

$$A\big(R^z_\psi R^y_{\theta - \frac{\pi}{2}} e_1\big) = \big(R^z_\psi R^y_{\theta - \frac{\pi}{2}} e_1\big) \qquad \text{so} \quad (R^y_{\frac{\pi}{2} - \theta} R^z_{-\psi} A R^z_\psi R^y_{\theta - \frac{\pi}{2}})e_1 = e_1 . \tag{1.6}$$

Since SO(3) is a group, hence closed under composition, we see that the product on the left is of the form R^x_α as in (1.5). Hence we can rearrange to find

$$A = R^z_\psi R^y_{\theta - \frac{\pi}{2}} R^x_\alpha R^y_{\frac{\pi}{2} - \theta} R^z_{-\psi} = R^{\hat{n}}_\alpha \tag{1.7}$$

and so we are done - Ψ is indeed surjective.

For continuity of Ψ, we first note that the sphere S^2 can be defined via a quotient $([0, \pi] \times [0, 2\pi))/\sim$ that identifies each point on the boundary of the rectangle to be equivalent. More explicitly, we first wrap the (θ, ψ) square into a cylinder, and then identify each point on the circular ends with a single point, given by the following equivalence relation

$$\begin{gathered} (\theta, 0) \sim (\theta, 2\pi) , \qquad (0, \psi) \sim (0, \psi') \quad \text{and} \quad (\pi, \psi) \sim (\pi, \psi') \\ \text{for all} \quad \theta \in [0, \pi], \ \ \psi, \psi' \in [0, 2\pi) . \end{gathered} \tag{1.8}$$

This equivalence relation can be shown to be homeomorphic to the sphere $S^2 = \{x \in \mathbb{R}^3 \ | \ \|x\| = 1\}$, but we leave this as an exercise to the reader as this is not a topology course.

We can then extend this equivalence on $(\theta, \psi) \in S^2$ to be one on $(\theta, \psi, \alpha) \in S^2 \times [0, 2\pi)$ denoted $\sim'$, which is merely $\sim$ on the first two components and the identity (i.e. for $\alpha_1, \alpha_2 \in [0, 2\pi)$, $\alpha_1 \sim' \alpha_2$ iff $\alpha_1 = \alpha_2$) in the third component. In doing this, we may appeal to the universal property of the quotient, which says in the case of the following diagram,

$$\begin{array}{ccccc} [0,\pi] \times [0,2\pi) \times [0,2\pi) & \xrightarrow{\ g\ } & \mathrm{SO}(3) & \hookrightarrow & M_3(\mathbb{C}) \cong \mathbb{C}^9 \\ \big\downarrow q & \nearrow f & & & \\ ([0,\pi] \times [0,2\pi) \times [0,2\pi))/\sim' & & & & \end{array} \tag{1.9}$$

if we have a continuous map g that preserves the equivalence relations (i.e. if $a \sim b$ implies $g(a) = g(b)$) then there exists a unique continuous map f such that the above diagram commutes. So we check: if $(\theta, 0, \alpha) \sim (\theta, 2\pi, \alpha)$ then

$$\begin{aligned} g(\theta, 0, \alpha) &= R^z_0 R^y_{\theta-\frac{\pi}{2}} R^x_\alpha R^y_{\frac{\pi}{2}-\theta} R^z_0 = R^y_{\theta-\frac{\pi}{2}} R^x_\alpha R^y_{\frac{\pi}{2}-\theta} \\ &= R^z_{2\pi} R^y_{\theta-\frac{\pi}{2}} R^x_\alpha R^y_{\frac{\pi}{2}-\theta} R^z_{-2\pi} = g(\theta, 2\pi, \alpha)\,. \end{aligned} \tag{1.10}$$

If $(0, \psi, \alpha) \sim (0, \psi', \alpha)$ then

$$\begin{aligned} g(0, \psi, \alpha) &= R^z_\psi R^y_{-\frac{\pi}{2}} R^x_\alpha R^y_{\frac{\pi}{2}} R^z_{-\psi} \\ &= \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} = R^z_\alpha = g(0, \psi', \alpha) \end{aligned} \tag{1.11}$$

and so we see there is no ψ dependence, hence we have $g(0, \psi, \alpha) = g(0, \psi', \alpha)$ (we would have provided the entire calculation to get to this point, but it simply did not fit in the margin - literally! I can provide photographic evidence if you don't believe me...). Finally we have the case $(\pi, \psi, \alpha) \sim (\pi, \psi', \alpha)$ where

$$\begin{aligned} g(\pi, \psi, \alpha) &= R^z_\psi R^y_{\frac{\pi}{2}} R^x_\alpha R^y_{-\frac{\pi}{2}} R^z_{-\psi} \\ &= \begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} = R^z_{-\alpha} = g(\pi, \psi', \alpha) \end{aligned} \tag{1.12}$$

and so once again with no ψ dependence we have our required property of g!

Before concluding that f in (1.9) is indeed continuous we need to ensure that g is continuous. But this is straight forward when SO(3) is endowed with the $\mathbb{C}^9$ topology as each entry is just a sum and product of trigonometric functions on the cuboid (with no singularities to worry about) and hence is clearly continuous. Therefore we see that f is continuous and so by the universal property of the quotient space we see that (1.1) is continuous and so we are done. □

Part b)

We may then define an equivalence relation on $S^2 \times [0, 2\pi)$ via $(\hat{n}, \alpha) \sim (\hat{m}, \beta)$ if $R^{\hat{n}}_\alpha = R^{\hat{m}}_\beta$. To give an explicit description of the relation $\sim$, suppose $R^{\hat{n}}_\alpha = R^m_\beta$. Intuitively we expect $\hat{n}$ to be the eigenvalue of $R^{\hat{n}}_\alpha$ but we should make sure. We may first observe from part a) that $\lambda = 1$ is guaranteed to be an eigenvalue of $R^{\hat{n}}_\alpha$ and so there is some vector $v \in \mathbb{R}^3$ such that $R^{\hat{n}}_\alpha v = v$. Suppose first that $\hat{n} = e_1$, hence this has the form

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} v_1 \\ \cos\alpha v_2 - \sin\alpha v_3 \\ \sin\alpha v_2 + \cos\alpha v_3 \end{pmatrix}\,. \tag{1.13}$$

Hence we see that v_1 is a free parameter, but in solving for v_2 from both equations we find

$$v_2 = \frac{\sin\alpha}{\cos\alpha - 1} v_3 \quad \text{and} \quad v_2 = \frac{1-\cos\alpha}{\sin\alpha} v_3\,. \tag{1.14}$$

In order for this to be consistent the coefficients must be equal, but if this was the case then we would have

$$\sin^2\alpha = -\cos^2\alpha + 2\cos\alpha - 1\,, \quad \text{so} \quad \cos\alpha = 0\,, \quad \text{so} \quad \alpha = \frac{\pi}{2}\,,\ \frac{3\pi}{2}\,. \tag{1.15}$$

But since we desire an eigenvector for all α, this is a contradiction and so we must have $v_2 = 0$ and $v_3 = 0$ (which is also true in the specific non-contradictory cases of α as solved above). Hence our eigenvector of $R^{e_1}_\alpha$ is $v = ke_1$ for some $k \in \mathbb{R}$, but then since we only care about unit vectors on the sphere we have $k = \pm 1$ and so $v = e_1$ or $v = -e_1$.

For the eigenvector w in the general case $R^{\hat{n}}_\alpha$ we have, using the invertibility of our rotation matrices,

$$R^{\hat{n}}_\alpha w = R^z_\psi R^y_{\theta-\frac{\pi}{2}} R^x_\alpha R^y_{\frac{\pi}{2}-\theta} R^z_{-\psi} w = w\,, \quad \text{so} \;\; R^x_\alpha(R^y_{\frac{\pi}{2}-\theta} R^z_{-\psi} w) = R^y_{\frac{\pi}{2}-\theta} R^z_{-\psi} w\,. \tag{1.16}$$

Hence we see that $R^y_{\frac{\pi}{2}-\theta} R^z_{-\psi} w$ is an eigenvector of R^x_α and so we must have

$$R^y_{\frac{\pi}{2}-\theta} R^z_{-\psi} w = \pm e_1 \quad \text{so} \;\; w = \pm R^z_\psi R^y_{\theta-\frac{\pi}{2}} e_1 = \pm\hat{n}\,. \tag{1.17}$$

Recall that the dimension of the eigenspace corresponding to $\lambda = 1$ is either 1 or 3 from part a). But if its dimension was 3 then $R^{\hat{n}}_\alpha$ would be the identity matrix and hence trivial, in other words if $\alpha = 0$ or $\beta = 0$ then nothing can be said about the relation of $\hat{n}$ and $\hat{m}$.

Supposing that we are in the non-trivial case, we must have a dimension 1 eigenspace. But then since $R^{\hat{n}}_\alpha = R^m_\beta$, it must be that $\hat{n} = \pm\hat{m}$!

In the first case, now suppose that $R^{\hat{n}}_\alpha = R^{\hat{n}}_\beta$ for some angles $\alpha, \beta \in [0, 2\pi)$ and some fixed unit vector $\hat{n}$. Then we have

$$\begin{aligned} R^z_\psi R^y_{\theta-\frac{\pi}{2}} R^x_\alpha R^y_{\frac{\pi}{2}-\theta} R^z_{-\psi} &= R^z_\psi R^y_{\theta-\frac{\pi}{2}} R^x_\beta R^y_{\frac{\pi}{2}-\theta} R^z_{-\psi}\,, \\ \text{so} \quad R^x_\alpha = (R^z_\psi R^y_{\theta-\frac{\pi}{2}})^{-1} R^z_\psi R^y_{\theta-\frac{\pi}{2}} R^x_\beta R^y_{\frac{\pi}{2}-\theta} R^z_{-\psi} &(R^y_{\frac{\pi}{2}-\theta} R^z_{-\psi})^{-1} = R^x_\beta\,. \end{aligned}$$

By comparison with the explicit matrix forms of R^x_α and R^x_β we see that we require $\cos\alpha = \cos\beta$ and $\sin\alpha = \sin\beta$ meaning that $\alpha = \beta \in [0, 2\pi)$.

In the case where $R^{\hat{n}}_\alpha = R^{-\hat{n}}_\beta$, the presence of the negative will simply reverse the orientation of rotation (easy to see with the right hand rule) which gives the simple relation

$$\alpha = 2\pi - \beta\,, \tag{1.18}$$

making suitable adjustments for principal arguments and such. Therefore we define the equivalence relation to be

$$(\hat{n}, \alpha) \sim (\hat{m}, \beta) \quad \text{if} \quad \begin{cases} \hat{n} = \hat{m} \quad \text{and} \quad \alpha = \beta \\ \hat{n} = -\hat{m} \quad \text{and} \quad \alpha = 2\pi - \beta \end{cases} . \quad \square \tag{1.19}$$

Q2. $C^\infty(W)$ is a sheaf

To define S^2 properly, we consider two homeomorphisms that parametrise the sphere in different ways, namely

$$\begin{aligned} j : (0,\pi)\times(0,2\pi) \to S^2\,, \qquad & j(\theta,\psi) = (\sin\theta\cos\psi, \sin\theta\sin\psi, \cos\theta) \\ \text{and} \quad j^{\text{alt}} : (0,\pi)\times(0,2\pi) \to S^2\,, \quad & j^{\text{alt}}(\theta',\psi') = (-\sin\theta'\cos\psi', \cos\theta', \sin\theta'\sin\psi') \end{aligned} \tag{2.1}$$

We may then define $U := \text{im}(j)$ and $V := \text{im}(j^{\text{alt}})$. Given an open set $W \subseteq S^2$, $C^\infty(W)$ denotes the set of all smooth functions, that is continuous functions $f : W \to \mathbb{R}$ such that $f|_{U\cap W} \in C^\infty_j(U\cap W)$ and $f|_{V\cap W} \in C^\infty_{j^{\text{alt}}}(V\cap W)$, where we define

$$\begin{aligned} C^\infty_j(U\cap W) &= \{f \in \text{Cts}(U\cap W, \mathbb{R}) \mid f\circ j \in C^\infty(j^{-1}(U\cap W)\}\,, \\ C^\infty_{j^{\text{alt}}}(V\cap W) &= \{f \in \text{Cts}(V\cap W, \mathbb{R}) \mid f\circ j^{\text{alt}} \in C^\infty(j^{\text{alt}-1}(V\cap W))\}\,. \end{aligned} \tag{2.2}$$

Here $C^\infty(j^{-1}A)$ refers to the usual smoothness in $\mathbb{R}^2$, that is, infinitely differentiable functions. Our goal is to show that $C^\infty(W)$ is a sheaf, so time to get gluing!

Part a)

Let $W' \subseteq W$ be open and let $f \in C^\infty(W)$, we want to show that $f|_{W'} \in C^\infty(W')$. First consider $(f|_{W'})\big|_{U\cap W'} = f|_{U\cap W'}$ which we want to show is in $C^\infty_j(U\cap W')$. It is clearly in $\text{Cts}(U\cap W')$ since $f \in C^\infty_j(U\cap W) \subseteq \text{Cts}(U\cap W, \mathbb{R})$ and $U\cap W' \subseteq U\cap W$. If $x \in j^{-1}(U\cap W')$, then $j(x) \in U\cap W' \subseteq U\cap W$ and so $x \in j^{-1}(U\cap W')$, hence $j^{-1}(U\cap W') \subseteq j^{-1}(U\cap W)$. Hence we have the following diagram

$$\underbrace{j^{-1}(U\cap W') \overset{\iota}{\hookrightarrow} j^{-1}(U\cap W) \xrightarrow{f|_{U\cap W}\circ j} \mathbb{R}}_{f|_{U\cap W'}\circ j}\,. \tag{2.3}$$

Clearly the inclusion ι is infinitely differentiable, and so by the chain rule and the fact that $f|_{U\cap W}\circ j$ is infinitely differentiable by assumption, we have that $f|_{U\cap W'}\circ j \in C^\infty(j^{-1}(U\cap W'))$ and so $f|_{U\cap W'} \in C^\infty_j(U\cap W')$. The same argument holds for j^{alt} and so $f|_{W'} \in C^\infty(W')$.

Part b)

Let $W \subseteq S^2$ be open and suppose $\{W_\alpha\}_{\alpha\in\Lambda}$ is an open cover of W (that is, for every $\alpha \in \Lambda$, W_α is an open subset of W and $\bigcup_{\alpha\in\Lambda} W_\alpha = W$), and let $\{f_\alpha\}_{\alpha\in\Lambda}$ be a family of functions $f_\alpha \in C^\infty(W_\alpha)$ such that

$$f_\alpha|_{W_\alpha\cap W_\beta} = f_\beta\big|_{W_\alpha\cap W_\beta} \quad \text{for all } \alpha,\beta \in \Lambda\,. \tag{2.4}$$

We will show that there exists a unique $f \in C^\infty(W)$ such that $f|_{W_\alpha} = f_\alpha$ for all $\alpha \in \Lambda$.

The essence of this question lies in the fact that smoothness is a local property that glues together exceptionally well due to its neighbourhood based definition. In other words, if smooth functions don't form a sheaf, then what hope do we have that anything else will?

As such, we can simply define a function $f : W \to \mathbb{R}$ that takes a given $x \in W$, which by the definition of our open cover must be contained in some W_α, and use the above gluing property to define

$$\text{for } x \in W_\alpha\,, \quad f(x) = f_\alpha(x)\,. \tag{2.5}$$

This is all well and good to write down, but clearly the trick is in showing f is well defined. It is clear that the codomain is indeed $\mathbb{R}$ as this is inherited from the f_α's. Suppose that we had some $x \in W_\alpha \cap W_\beta$ for some $\alpha \neq \beta \in \Lambda$, meaning we could have $f(x) = f_\alpha(x)$ or alternatively $f(x) = f_\beta(x)$. But by the gluing property we have

$$f_\alpha(x) = f_\alpha\big|_{W_\alpha \cap W_\beta}(x) = f_\beta\big|_{W_\alpha \cap W_\beta}(x) = f_\beta(x) \tag{2.6}$$

thus showing that either of the two "possible" definitions of $f(x)$ for this given x value agree, hence $f(x)$ itself is well defined. It is also clear by our definition that $f|_{W_\alpha} = f_\alpha$ for any $\alpha \in \Lambda$. To see that f is unique, suppose there was another $g : W \to \mathbb{R}$ such that $g|_{W_\alpha} = f_\alpha$ for all $\alpha \in \Lambda$. Then for some $x \in W_\alpha \subseteq W$ we have

$$g(x) = g|_{W_\alpha}(x) = f_\alpha(x) = f(x) \tag{2.7}$$

thus showing that $f = g$ and so f is unique. It remains to show that f is in $C^\infty(W)$.

We are given that each $f_\alpha \in C^\infty(W_\alpha)$, which is to say that $f_\alpha \circ j \in C^\infty(j^{-1}(W_\alpha \cap U))$ as above. That is, $(f_\alpha \circ j)\big|_{j^{-1}(W_\alpha \cap U)}$ is a smooth function. But by definition we have

$$(f \circ j)\big|_{j^{-1}(W_\alpha \cap U)} = f|_{W_\alpha \cap U} \circ j|_{j^{-1}(W_\alpha \cap U)} = f_\alpha|_{W_\alpha \cap U} \circ j|_{j^{-1}(W_\alpha \cap U)} = (f_\alpha \circ j)\big|_{j^{-1}(W_\alpha \cap U)}\,. \tag{2.8}$$

So far this is only defined on W_α, whereas we require $f \circ j \in C^\infty(j^{-1}(W \cap U))$. But since $W = \bigcup_{\alpha \in \Lambda} W_\alpha$ we have

$$f|_{W \cap U} = f|_{(\cup_\alpha W_\alpha) \cap U} = f|_{\cup_\alpha (W_\alpha \cap U)}\,. \tag{2.9}$$

Given an $x \in W_\beta \cap U$ for some $\beta \in \Lambda$ we may then take some open neighbourhood $A \subseteq W_\beta$ about x and write

$$f|_{A \cap (\cup_\alpha (W_\alpha \cap U))} = f|_{A \cap W_\beta \cap U}\,, \tag{2.10}$$

where we note the same well-defined property holds as above allowing us to just take one W_β. Thus, applying (2.8) we have

$$(f \circ j)\big|_{j^{-1}(W \cap U \cap A)} = (f_\beta \circ j)\big|_{j^{-1}(W_\beta \cap U \cap A)} \tag{2.11}$$

for some $\beta \in \Lambda$. Since the right hand side is smooth at x by our assumption on $\{f_\alpha\}_\alpha$ we have smoothness at each point $x \in W$. That is, $f \circ j \in C^\infty(j^{-1}(W \cap U))$. We can apply the exact same logic to the j^{alt} based conditions and thus we have that our chosen f is in $C^\infty(W)$. Therefore $C^\infty(W)$ is a sheaf. $\square$

Q3. Laplace vs Laplace

Let f be a smooth function on $\mathbb{R}^3$. We want to prove the following Laplacian identity

$$\Delta_{\mathbb{R}^3} f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\Delta_{S^2} f =: Lf \tag{3.1}$$

where we define

$$\Delta_{\mathbb{R}^3} f := \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) f \,, \tag{3.2}$$

$$\text{and} \quad \Delta_{S^2} f := \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial f}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 f}{\partial\psi^2} \,, \tag{3.3}$$

under the following parameterisation in spherical coordinates:

$$x(r,\theta,\psi) = r\sin\theta\cos\psi \,, \quad y(r,\theta,\psi) = r\sin\theta\sin\psi \,, \quad z(r,\theta,\psi) = r\cos\theta \tag{3.4}$$
$$\text{where} \quad r \in [0,\infty) \,, \quad \theta \in [0,\pi] \,, \quad \psi \in [0,2\pi) \,.$$

We first calculate basic derivatives of the above coordinates,

$$\begin{aligned}
&\frac{\partial x}{\partial r} = \sin\theta\cos\psi && \frac{\partial y}{\partial r} = \sin\theta\sin\psi && \frac{\partial z}{\partial r} = \cos\theta \\
&\frac{\partial x}{\partial \theta} = r\cos\theta\cos\psi && \frac{\partial y}{\partial \theta} = r\cos\theta\sin\psi && \frac{\partial z}{\partial \theta} = -r\sin\theta \\
&\frac{\partial x}{\partial \psi} = -r\sin\theta\sin\psi && \frac{\partial y}{\partial \psi} = r\sin\theta\cos\psi && \frac{\partial z}{\partial \psi} = 0 \,.
\end{aligned} \tag{3.5}$$

To calculate $\frac{\partial f}{\partial x_i}$ for each $x_i = r\,,\theta\,,\psi$ we can appeal to the chain rule for multivariable functions, namely

$$\frac{\partial}{\partial x_i} f\big(x(r,\theta,\psi), y(r,\theta,\psi), z(r,\theta,\psi)\big) = \frac{\partial f}{\partial x}\frac{\partial x}{\partial x_i} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial x_i} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial x_i} \,. \tag{3.6}$$

Thus we can begin the painful process of calculating the many terms in the expression (3.1). Firstly we have

$$\begin{aligned}
\frac{\partial^2 f}{\partial\psi^2} &= \frac{\partial}{\partial\psi}\left(-\frac{\partial f}{\partial x} r\sin\theta\sin\psi + \frac{\partial f}{\partial y} r\sin\theta\cos\psi\right) \\
&= -\frac{\partial f}{\partial x}(r\sin\theta\cos\psi) - (r\sin\theta\sin\psi)\left[\frac{\partial^2 f}{\partial x^2}(-r\sin\theta\sin\psi) + \frac{\partial^2 f}{\partial y\partial x} r\sin\theta\cos\psi\right] \\
&\quad - \frac{\partial f}{\partial y}(r\sin\theta\sin\psi) + (r\sin\theta\cos\psi)\left[\frac{\partial^2 f}{\partial x\partial y}(-r\sin\theta\sin\psi) + \frac{\partial^2 f}{\partial y^2} r\sin\theta\cos\psi\right] \\
&= -r\sin\theta\cos\psi\frac{\partial f}{\partial x} - r\sin\theta\sin\psi\frac{\partial f}{\partial y} + r^2\sin^2\theta\sin^2\psi\frac{\partial^2 f}{\partial x^2} + r^2\sin^2\theta\cos^2\psi\frac{\partial^2 f}{\partial y^2} \\
&\quad - 2r^2\sin^2\theta\sin\psi\cos\psi\frac{\partial^2 f}{\partial y\partial x} \,,
\end{aligned} \tag{3.7}$$

. . . and this was the easy one.

We then turn to

$$
\begin{aligned}
\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) &= \frac{\partial}{\partial\theta}\left(\sin\theta\left[\frac{\partial f}{\partial x}(r\cos\theta\cos\psi)+\frac{\partial f}{\partial y}(r\cos\theta\sin\psi)-\frac{\partial f}{\partial z}r\sin\theta\right]\right)\\
&= r\cos 2\theta\cos\psi\frac{\partial f}{\partial x}+r\cos 2\theta\sin\psi\frac{\partial f}{\partial y}-2\sin\theta\cos\theta\frac{\partial f}{\partial z}\\
&\quad + r\sin\theta\cos\theta\cos\psi\left[\frac{\partial^2 f}{\partial x^2}(r\cos\theta\cos\psi)+\frac{\partial^2 f}{\partial y\partial x}(r\cos\theta\sin\psi)-\frac{\partial^2 f}{\partial z\partial x}r\sin\theta\right]\\
&\quad + r\sin\theta\cos\theta\sin\psi\left[\frac{\partial^2 f}{\partial x\partial y}(r\cos\theta\cos\psi)+\frac{\partial^2 f}{\partial y^2}(r\cos\theta\sin\psi)-\frac{\partial^2 f}{\partial z\partial y}r\sin\theta\right]\\
&\quad - r\sin^2\theta\left[\frac{\partial^2 f}{\partial x\partial z}(r\cos\theta\cos\psi)+\frac{\partial^2 f}{\partial y\partial z}(r\cos\theta\sin\psi)-\frac{\partial^2 f}{\partial z^2}r\sin\theta\right]\\
&= r\cos 2\theta\cos\psi\frac{\partial f}{\partial x}+r\cos 2\theta\sin\psi\frac{\partial f}{\partial y}-2r\sin\theta\cos\theta\frac{\partial f}{\partial z}\\
&\quad + r^2\sin\theta\cos^2\theta\cos^2\psi\frac{\partial^2 f}{\partial x^2}+r^2\sin\theta\cos^2\theta\sin^2\psi\frac{\partial^2 f}{\partial y^2}+r^2\sin^3\theta\frac{\partial^2 f}{\partial z^2}\\
&\quad + 2r^2\sin\theta\cos^2\theta\sin\psi\cos\psi\frac{\partial^2 f}{\partial x\partial y}-2r^2\sin^2\theta\cos\theta\cos\psi\frac{\partial^2 f}{\partial x\partial z}\\
&\quad - 2r^2\sin^2\theta\cos\theta\sin\psi\frac{\partial^2 f}{\partial y\partial z}\,. \qquad (3.8)
\end{aligned}
$$

Through gritted teeth, we then finally have

$$
\begin{aligned}
\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) &= \frac{\partial}{\partial r}\left(r^2\sin\theta\cos\psi\frac{\partial f}{\partial x}+r^2\sin\theta\sin\psi\frac{\partial f}{\partial y}+r^2\cos\theta\frac{\partial f}{\partial z}\right)\\
&= 2r\sin\theta\cos\psi\frac{\partial f}{\partial x}+2r\sin\theta\sin\psi\frac{\partial f}{\partial y}+2r\cos\theta\frac{\partial f}{\partial z}\\
&\quad + r^2\sin\theta\cos\psi\left[\sin\theta\cos\psi\frac{\partial^2 f}{\partial x^2}+\sin\theta\sin\psi\frac{\partial^2 f}{\partial y\partial x}+\cos\theta\frac{\partial^2 f}{\partial z\partial x}\right]\\
&\quad + r^2\sin\theta\sin\psi\left[\sin\theta\cos\psi\frac{\partial^2 f}{\partial x\partial y}+\sin\theta\sin\psi\frac{\partial^2 f}{\partial y^2}+\cos\theta\frac{\partial^2 f}{\partial z\partial y}\right]\\
&\quad + r^2\cos\theta\left[\sin\theta\cos\psi\frac{\partial^2 f}{\partial x\partial z}+\sin\theta\sin\psi\frac{\partial^2 f}{\partial y\partial z}+\cos\theta\frac{\partial^2 f}{\partial z^2}\right]\\
&= 2r\sin\theta\cos\psi\frac{\partial f}{\partial x}+2r\sin\theta\sin\psi\frac{\partial f}{\partial y}+2r\cos\theta\frac{\partial f}{\partial z}\\
&\quad + r^2\sin^2\theta\cos^2\psi\frac{\partial^2 f}{\partial x^2}+r^2\sin^2\theta\sin^2\psi\frac{\partial^2 f}{\partial y^2}+r^2\cos^2\theta\frac{\partial^2 f}{\partial z^2}\\
&\quad + 2r^2\sin^2\theta\sin\psi\cos\psi\frac{\partial^2 f}{\partial x\partial y}+2r^2\sin\theta\cos\theta\cos\psi\frac{\partial^2 f}{\partial x\partial z}\\
&\quad + 2r^2\sin\theta\cos\theta\sin\psi\frac{\partial^2 f}{\partial y\partial z}\,. \qquad (3.9)
\end{aligned}
$$

Well that took a lot of blood sweat and tears. Putting all of this together and performing the necessary divisions we thus have

$$
\begin{aligned}
Lf &= \frac{2}{r}\sin\theta\cos\psi\frac{\partial f}{\partial x} + \frac{2}{r}\sin\theta\sin\psi\frac{\partial f}{\partial y} + \frac{2}{r}\cos\theta\frac{\partial f}{\partial z} + \sin^2\theta\cos^2\psi\frac{\partial^2 f}{\partial x^2} + \sin^2\theta\sin^2\psi\frac{\partial^2 f}{\partial y^2} + \cos^2\theta\frac{\partial^2 f}{\partial z^2} \\
&\quad + 2\sin^2\theta\sin\psi\cos\psi\frac{\partial^2 f}{\partial x\partial y} + 2\sin\theta\cos\theta\cos\psi\frac{\partial^2 f}{\partial x\partial z} + 2\sin\theta\cos\theta\sin\psi\frac{\partial^2 f}{\partial y\partial z} \\
&\quad + \frac{\cos 2\theta\cos\psi}{r\sin\theta}\frac{\partial f}{\partial x} + \frac{\cos 2\theta\sin\psi}{r\sin\theta}\frac{\partial f}{\partial y} - \frac{2\cos\theta}{r}\frac{\partial f}{\partial z} + \cos^2\theta\cos^2\psi\frac{\partial^2 f}{\partial x^2} + \cos^2\theta\sin^2\psi\frac{\partial^2 f}{\partial y^2} + \sin^2\theta\frac{\partial^2 f}{\partial z^2} \\
&\quad + 2\cos^2\theta\sin\psi\cos\psi\frac{\partial^2 f}{\partial x\partial y} - 2\sin\theta\cos\theta\cos\psi\frac{\partial^2 f}{\partial x\partial z} - 2\sin\theta\cos\theta\sin\psi\frac{\partial^2 f}{\partial y\partial z} \\
&\quad - \frac{\cos\psi}{r\sin\theta}\frac{\partial f}{\partial x} - \frac{\sin\psi}{r\sin\theta}\frac{\partial f}{\partial y} + \sin^2\psi\frac{\partial^2 f}{\partial x^2} + \cos^2\psi\frac{\partial^2 f}{\partial y^2} - 2\sin\psi\cos\psi\frac{\partial^2 f}{\partial y\partial x} \\
&= \frac{2\sin^2\theta\cos\psi + (\cos^2\theta - \sin^2\theta)\cos\psi - \cos\psi}{r\sin\theta}\frac{\partial f}{\partial x} + \frac{2\sin^2\theta\sin\psi + (\cos^2\theta - \sin^2\theta)\sin\psi - \sin\psi)}{r\sin\theta}\frac{\partial f}{\partial y} \\
&\quad + \frac{2\cos\theta - 2\cos\theta}{r}\frac{\partial f}{\partial z} + \left(2\sin^2\theta\sin\psi\cos\psi + 2\cos^2\theta\sin\psi\cos\psi - 2\sin\psi\cos\psi\right)\frac{\partial^2 f}{\partial x\partial y} \\
&\quad + \left(2\sin\theta\cos\theta\cos\psi - 2\sin\theta\cos\theta\cos\psi\right)\frac{\partial^2 f}{\partial x\partial z} + \left(2\sin\theta\cos\theta\sin\psi - 2\sin\theta\cos\theta\sin\psi\right)\frac{\partial f}{\partial y\partial z} \\
&\quad + \left(\sin^2\theta\cos^2\psi + \cos^2\theta\cos^2\psi + \sin^2\psi\right)\frac{\partial^2 f}{\partial x^2} + \left(\sin^2\theta\sin^2\psi + \cos^2\theta\sin^2\psi + \cos^2\psi\right)\frac{\partial^2 f}{\partial y^2} \\
&\quad + \left(\cos^2\theta + \sin^2\theta\right)\frac{\partial^2 f}{\partial z^2} \\
&= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}. \qquad (3.10)
\end{aligned}
$$

The last equality is due to repeated use of the Pythagorean identity, causing most terms to be 0. We offer praise to the mathematical gods that no arithmetic mistakes foiled us and we have indeed arrived at the identity promised in (3.1). Curse you, Mr Lecturer, curse you... □

Lecture 4

Q4. Families of Operators

Let V be a finite dimensional vector space with $d = \dim_{\mathbb{C}}(V)$, and let $\mathrm{End}_{\mathbb{C}}(V)$ denote the $\mathbb{C}$-vector space of linear operators on V. Given an ordered basis β of V there is an isomorphism of vector spaces sending an operator to its matrix, that is

$$C_\beta : \mathrm{End}_{\mathbb{C}}(V) \xrightarrow{\cong} M_d(V)\,, \quad C_\beta(T) = [T]_\beta^\beta\,. \tag{4.1}$$

For an open subset $U \subseteq \mathbb{R}^n$ we say that a function $f : U \to \mathrm{End}_{\mathbb{C}}(V)$ is smooth, that is $f \in C^\infty(U, \mathrm{End}_{\mathbb{C}}(V))$, if the composite

$$U \xrightarrow{f} \mathrm{End}_{\mathbb{C}}(V) \xrightarrow{C_\beta} M_d(\mathbb{C}) \xrightarrow{\cong} \mathbb{C}^{d^2} \tag{4.2}$$

is smooth, in other words, the entries of the matrix $[f(u)]_\beta^\beta$ are smooth functions of u.

We may then define for $1 \leq i \leq n$ a $\mathbb{C}$-linear operator $\frac{\partial}{\partial x_i} : C^\infty(U, \mathrm{End}_{\mathbb{C}}(V)) \to C^\infty(U, \mathrm{End}_{\mathbb{C}}(V))$ to be the following composite

$$C^\infty(U, \mathrm{End}_{\mathbb{C}}(V)) \xrightarrow{C_\beta \circ (-)} C^\infty(U, M_d(\mathbb{C})) \xrightarrow{\frac{\partial}{\partial x_i}} C^\infty(U, M_d(\mathbb{C})) \xrightarrow{C_\beta^{-1} \circ (-)} C^\infty(U, \mathrm{End}_{\mathbb{C}}(V)) \tag{4.3}$$

where $\frac{\partial}{\partial x_i}$ acts on matrices of functions entry-wise, that is, if $f : U \to M_d(\mathbb{C})$ is identified with a matrix $(f_{jk}(u))$ of functions then the derivative is $(\frac{\partial}{\partial x_i}(f_{jk}))$. Now that the definitions are out of the way, let's get down to business.

Part a)

In all of the notation as above, suppose $f : U \to \mathrm{End}_{\mathbb{C}}(V)$ is smooth with respect to some basis β of V, and let β' be some other basis of V. Since V is a finite dimensional vector space, we may find a change of basis matrix $P \in M_d(\mathbb{C})$ that takes a vector $u \in V$ expressed in terms of the basis β and produces the same vector in terms of β', that is,

$$[u]_{\beta'} = [P]_\beta^{\beta'} [u]_\beta\,. \tag{4.4}$$

Such a matrix P is deterministic with constant entries $p_{jk} \in \mathbb{C}$. Hence we may express

$$[f(u)]_{\beta'}^{\beta'} = P^{-1} [f(u)]_\beta^\beta P \tag{4.5}$$

where the entries of $[f(u)]_{\beta'}^{\beta'}$ are $\mathbb{C}$-linear combinations of those in $[f(u)]_\beta^\beta$. Scalar multiplication and addition are smooth functions, hence the entries of $[f(u)]_{\beta'}^{\beta'}$ are smooth functions of u and so f is smooth with respect to the other arbitrary basis β'. Hence f is smooth with respect to some basis β if and only if it is smooth with respect to any basis. □

Part b)

We can then show that $\frac{\partial}{\partial x_i} : C^\infty(U, \mathrm{End}_{\mathbb{C}}(V)) \to C^\infty(U, \mathrm{End}_{\mathbb{C}}(V))$ is independent of the basis β used to define it. Suppose we had defined $\frac{\partial}{\partial x_i}$ via some other basis β' of V, then by (4.5) the composite in (4.3) for $f \in C^\infty(U, \mathrm{End}_{\mathbb{C}}(V))$ becomes

$$\begin{aligned} C_{\beta'}^{-1}\left(\frac{\partial}{\partial x_i}(C_{\beta'}(f))\right) &= C_{\beta'}^{-1}\left(\frac{\partial}{\partial x_i}(P^{-1}[f(u)]_\beta^\beta P)\right) = C_{\beta'}^{-1}\left(P^{-1}\frac{\partial}{\partial x_i}[f(u)]_\beta^\beta P\right) \\ &= C_{\beta'}^{-1}\left(\left[\frac{\partial}{\partial x_i}f(u)\right]_{\beta'}^{\beta'}\right) = \frac{\partial}{\partial x_i}f(u) =: C_\beta^{-1}\left(\frac{\partial}{\partial x_i}(C_\beta(f))\right). \end{aligned} \tag{4.6}$$

Here $\frac{\partial}{\partial x_i}f(u)$ refers to the element of $C^\infty(U, \mathrm{End}_{\mathbb{C}}(V))$. In the second equality we have used the linearity of $\frac{\partial}{\partial x_i}$ since P is deterministic as before. Hence we see that $\frac{\partial}{\partial x_i}$ is independent of the choice of basis on V and is thus well defined on $C^\infty(U, \mathrm{End}_{\mathbb{C}}(V))$. □

Part c)

We may show that the vector space $C^\infty(U, \mathrm{End}_{\mathbb{C}}(V))$ is a $\mathbb{C}$-algebra by defining a binary operation $C^\infty(U, \mathrm{End}_{\mathbb{C}}(V)) \times C^\infty(U, \mathrm{End}_{\mathbb{C}}(V)) \to C^\infty(U, \mathrm{End}_{\mathbb{C}}(V))$ as

$$(fg)(u) := f(u) \circ g(u) \cong F(u)G(u). \tag{4.7}$$

For the purposes of this question we may view $f(u)$ and $g(u)$ in their isomorphic matrix forms where $C_\beta(f(u)) = F = F(u)$ and similarly for G and H below, meaning that this composition is merely matrix multiplication, i.e. $C_\beta(f(u) \circ g(u)) = FG$. To see that the operation is well defined, that is, the composite is indeed an element of $C^\infty(U, \mathrm{End}_{\mathbb{C}}(V))$, we note that

$$[FG]_{ij} = \sum_{k=1}^{n} F_{ik}(u)G_{kj}(u) \tag{4.8}$$

and so since each $F_{ik}(u)$ and $G_{kj}(u)$ is a smooth function of u by definition, the linear combination of products $F_{ik}(u)G_{kj}(u)$ will also be smooth, hence the binary operation is well defined.

We can then show that our operation is left distributitive: let $f, g, h \in C^\infty(U, \mathrm{End}_{\mathbb{C}}(V))$ and $u \in U$, then

$$\begin{aligned} ((f+g)h)(u) &= (f+g)(u) \circ h(u) = (f(u) + g(u)) \circ h(u) \\ &\cong (F+G)H = FH + GH \cong (f(u) \circ h(u)) + (g(u) \circ h(u)), \end{aligned} \tag{4.9}$$

thus the operation is left distributitive. Right distributitivity follows immediately from an identical calculation. Thus we just need to show compatibility with scalars: let $\zeta, \gamma \in \mathbb{C}$ and f, g as before, then we have

$$\begin{aligned} ((\zeta f)(\gamma g))(u) &= (\zeta f)(u) \circ (\gamma g)(u) \cong (\zeta F)(\gamma G) \\ &= \zeta\gamma FG \cong (\zeta\gamma)(f(u) \circ g(u)) = (\zeta\gamma)(fg)(u), \end{aligned} \tag{4.10}$$

hence showing compatibility with scalars. Thus, $C^\infty(U, \mathrm{End}_{\mathbb{C}}(V))$ is a $\mathbb{C}$-algebra due to inheritance of the properties of matrix multiplication. □

Part d)

Let $f, g \in C^\infty(U, \mathrm{End}_{\mathbb{C}}(V))$. We want to show that $\frac{\partial}{\partial x_i}$ satisfies the Leibniz rule, that is,

$$\frac{\partial}{\partial x_i}((fg)(u)) = \frac{\partial f(u)}{\partial x_i}g(u) + f(u)\frac{\partial g(u)}{\partial x_i}. \tag{4.11}$$

We calculate

$$\frac{\partial}{\partial x_i}((fg)(u)) = \frac{\partial}{\partial x_i}(f(u)\circ g(u)) = \frac{\partial}{\partial x_i}\big(F(u)G(u))$$

and so for a given entry (j,k) we have

$$\begin{aligned}\frac{\partial}{\partial x_i}\big[F(u)G(u)\big]_{jk} &= \frac{\partial}{\partial x_i}\sum_{h=1}^{n} F_{jh}(u)G_{hk}(u)\\ &= \sum_{h=1}^{n}\left[\frac{\partial F_{jh}(u)}{\partial x_i}G_{hk}(u) + F_{jh}(u)\frac{\partial G_{hk}(u)}{\partial x_i}\right]\\ &= \left[\frac{\partial F(u)}{\partial x_i}G(u)\right]_{jk} + \left[F(u)\frac{\partial G(u)}{\partial x_i}\right]_{jk}\\ &= \left[\frac{\partial F(u)}{\partial x_i}G(u) + F(u)\frac{\partial G(u)}{\partial x_i}\right]_{jk}. \end{aligned} \tag{4.12}$$

Note that the second equality follows from linearity and the product rule. Since $\frac{\partial}{\partial x_i}$ acts on a matrix piecewise as explained in part b), we have

$$\frac{\partial}{\partial x_i}\big[F(u)G(u)\big] = \frac{\partial F(u)}{\partial x_i}G(u) + F(u)\frac{\partial G(u)}{\partial x_i} \tag{4.13}$$

and so identifying this with (4.11) we are done. □

Part e)

Let $f \in C^\infty(U, \mathrm{End}_{\mathbb{C}}(V))$ and let β be a fixed basis of V. We can define the trace and determinant of a matrix $A = [A]_\beta^\beta$ as

$$\begin{aligned} &\mathrm{tr} : M_d(\mathbb{C}) \longrightarrow \mathbb{C} && \det : M_d(\mathbb{C}) \longrightarrow \mathbb{C}\\ &\mathrm{tr}(A) = \sum_{i=1}^{d} a_{ii} && \det(A) = \sum_{\sigma\in S_n}\left(\mathrm{sgn}(\sigma)\prod_{i=1}^{d} a_{i,\sigma_i}\right). \end{aligned} \tag{4.14}$$

So, to extend these to acting on an endomorphism $f(u)$ we simply apply the same idea as in the definition of the derivative in (4.3) and write

$$\begin{aligned} &\mathrm{tr}^\beta_{\mathrm{End}} : \mathrm{End}_{\mathbb{C}}(V) \longrightarrow \mathbb{C} && \det{}^\beta_{\mathrm{End}}(f(u)) : \mathrm{End}_{\mathbb{C}}(V) \longrightarrow \mathbb{C}\\ &\mathrm{tr}^\beta_{\mathrm{End}}(f(u)) = \mathrm{tr}\left(\big[f(u)\big]^\beta_\beta\right) \quad\text{and} && \det{}^\beta_{\mathrm{End}}(f(u)) = \det\left(\big[f(u)\big]^\beta_\beta\right). \end{aligned} \tag{4.15}$$

We may then show, as in earlier parts, that this choice of definition is independent of a choice of basis. Here we recall two key properties of the trace and determinant for square matrices $A, B, C \in M_d(\mathbb{C})$,

$$\mathrm{tr}(ABC) = \mathrm{tr}(BCA) = \mathrm{tr}(CAB) \quad\text{and}\quad \det(ABC) = \det(A)\det(B)\det(C). \tag{4.16}$$

Suppose we had defined $\mathrm{tr}_{\mathrm{End}}$ using another basis $\beta' \neq \beta$ of V instead, then using the cyclicity property above and the change of basis formula again from (4.5) we have

$$\begin{aligned}\mathrm{tr}^{\beta'}_{\mathrm{End}}(f(u)) &= \mathrm{tr}(\big[f(u)\big]^{\beta'}_{\beta'}) = \mathrm{tr}\left(P^{-1}\big[f(u)\big]^\beta_\beta P\right)\\ &= \mathrm{tr}(\big[f(u)\big]^\beta_\beta PP^{-1}) = \mathrm{tr}(\big[f(u)\big]^\beta_\beta) = \mathrm{tr}^\beta_{\mathrm{End}}(f(u)), \end{aligned} \tag{4.17}$$

hence we see the definition is independent of the choice of basis. Similarly for the determinant we have

$$\begin{aligned}\det^{\beta'}_{\text{End}}(f(u)) &= \det(\left[f(u)\right]^{\beta'}_{\beta'}) = \det\left(P^{-1}\left[f(u)\right]^{\beta}_{\beta}P\right)\\ &= \frac{\det\left(\left[f(u)\right]^{\beta}_{\beta}\right)\det P}{\det P} = \det\left(\left[f(u)\right]^{\beta}_{\beta}\right) = \det^{\beta}_{\text{End}}(f(u))\,,\end{aligned} \tag{4.18}$$

and so once again this definition is independent of the choice of basis. Thus as functions $U \to \mathbb{C}$ we see that for some basis β such that $\left(\left[f(u)\right]^{\beta}_{\beta}\right)_{i,j} = f_{ij}(u)$ we have

$$u \mapsto \text{tr}^{\beta}_{\text{End}}(f(u)) = \sum_{i=1}^{d} f_{ii}(u)\,, \tag{4.19}$$

$$\text{and} \quad u \mapsto \det^{\beta}_{\text{End}}(f(u)) = \sum_{\sigma\in S_n}\left(\text{sgn}(\sigma)\prod_{i=1}^{d} f_{i,\sigma_i}(u)\right)$$

are both smooth functions as they are just sums and products of the components $f_{ij}(u)$, which are all smooth themselves, hence the sums and products are smooth and so the trace and determinant of an endomoprhism $f(u)$ are well defined and smooth in u and so we are done. □

Q5. Suspicious looking formula

In Theorem L4-5 we have proven that the map

$$\begin{gathered}\rho : \text{SO}(3)^{\text{op}} \longrightarrow \text{Aut}_{\mathbb{C}}(\mathcal{P}_k(3)) \\ \rho(R^{\hat{n}}_{\alpha}) = \exp\left(\alpha\left[t_2\frac{\partial}{\partial t_3} - t_3\frac{\partial}{\partial t_2}\right]\right) = \sum_{n=0}^{\infty}\frac{\alpha^n}{n!}\left[t_2\frac{\partial}{\partial t_3} - t_3\frac{\partial}{\partial t_2}\right]^n\end{gathered} \tag{5.1}$$

is well defined and consistent with previous definitions of $\rho(R^{\hat{n}}_{\alpha})$ as operators on $\mathcal{P}_k(3)$. Note that t_1, t_2, t_3 refer to the standard x, y, z coordinates being rotated (with positive orientation) such that $\hat{n}$ is the x-axis. Here we will attempt to make sense of the suspicious looking formula that arises from this,

$$1 = \rho(R^{\hat{n}}_{2\pi}) = \exp\left(2\pi\left[t_2\frac{\partial}{\partial t_3} - t_3\frac{\partial}{\partial t_2}\right]\right)\,, \tag{5.2}$$

as an operator on $\mathcal{P}_k(3)$. Before we do, we may calculate provide a new set of coordinates for t_1, t_2, t_3 to make the differential operator in (5.1) more clear. To this end, define new (cylindrical) coordinates (h, r, θ) such that

$$t_1 = h, \quad t_2 = r\cos\theta, \quad t_3 = r\sin\theta \quad \text{for } h\in\mathbb{R},\ r\in[0,\infty), \text{ and } \theta\in[0,2\pi)\,. \tag{5.3}$$

Note that this gives rise to inverse coordinates of

$$h = t_1\,, \quad r = \sqrt{t_2^2 + t_3^2}\,, \quad \theta = \arctan\left(\frac{t_3}{t_2}\right)\,, \tag{5.4}$$

where appropriate care should be taken for domain of the polar coordinate, but for the purposes of this we will just take arctan which will give the essence of the answer here. We then have for $f = f\big(h(t_1,t_2,t_3), r(t_1,t_2,t_3), \theta(t_1,t_2,t_3)\big)$

$$\frac{\partial f}{\partial t_2} = \frac{\partial f}{\partial h}\frac{\partial h}{\partial t_2} + \frac{\partial f}{\partial r}\frac{\partial r}{\partial t_2} + \frac{\partial f}{\partial \theta}\frac{\partial \theta}{\partial t_2} = \frac{t_2}{\sqrt{t_2^2+t_3^2}}\frac{\partial f}{\partial r} - \frac{t_3}{t_2^2+t_3^2}\frac{\partial f}{\partial \theta} = \cos\theta\frac{\partial f}{\partial r} - \frac{\sin\theta}{r}\frac{\partial f}{\partial \theta}\,, \tag{5.5}$$

and then

$$\frac{\partial f}{\partial t_3} = \frac{\partial f}{\partial h}\frac{\partial h}{\partial t_3} + \frac{\partial f}{\partial r}\frac{\partial r}{\partial t_3} + \frac{\partial f}{\partial \theta}\frac{\partial \theta}{\partial t_3} = \frac{t_3}{\sqrt{t_2^2+t_3^2}}\frac{\partial f}{\partial r} + \frac{t_2}{t_2^2+t_3^2}\frac{\partial f}{\partial \theta} = \sin\theta\frac{\partial f}{\partial r} + \frac{\cos\theta}{r}\frac{\partial f}{\partial \theta}\,. \tag{5.6}$$

Thus as an operator we have

$$t_2\frac{\partial}{\partial t_3} - t_3\frac{\partial}{\partial t_2} = r\cos\theta\sin\theta\frac{\partial}{\partial r} + \cos^2\theta\frac{\partial}{\partial\theta} - r\sin\theta\cos\theta\frac{\partial}{\partial r} + \sin^2\theta\frac{\partial}{\partial\theta} = \frac{\partial}{\partial\theta}\,. \tag{5.7}$$

How convenient! Now recall that an arbitrary $P(t_1,t_2,t_3)\in\mathcal{P}_k(3)$ has the form

$$\begin{aligned} P(t_1,t_2,t_3) &= \sum_{|\beta|=k} c_\beta t^\beta = \sum_{\beta_1+\beta_2+\beta_3=k} c_\beta t_1^{\beta_1}t_2^{\beta_2}t_3^{\beta_3} \\ &= \sum_{\beta_1+\beta_2+\beta_3=k} c_\beta h^{\beta_1} r^{\beta_2+\beta_3}\cos^{\beta_2}\theta\sin^{\beta_3}\theta = P(h,r,\theta)\,, \end{aligned} \tag{5.8}$$

where $c_\beta\in\mathbb{C}$. Importantly, this is a *finite* sum, since we know that $\dim_{\mathbb{C}}(\mathcal{P}_k(3)) = \binom{k+2}{k}$ for any fixed k. This tells us that any $P(h,r,\theta)$ is an *analytic function* in θ since it is just a polynomial in smooth trigonmetric functions, hence we may write it as a Taylor series about $\theta=0$, namely

$$P(h,r,\theta) = \sum_{m=0}^{\infty} Q_m(r,h)\theta^m \tag{5.9}$$

where $Q_m(r,h)$ are some analytic functions. Thus we see that

$$\rho(R^{\hat{n}}_\alpha)(P) = \sum_{n=0}^{\infty}\frac{\alpha^n}{n!}\frac{\partial^n}{\partial\theta^n}\sum_{m=0}^{\infty}Q_m(r,h)\theta^m\,. \tag{5.10}$$

Before expanding this out, we first perform the simple calculation

$$\frac{\partial^n}{\partial\theta^n}\theta^m = \begin{cases} \frac{m!}{(m-n)!}\theta^{m-n} & \text{if} \quad n\le m \\ 0 & \text{if} \quad n>m \end{cases} \tag{5.11}$$

where we note that $\frac{m!}{(m-n)!} = n!\binom{m}{n}$. So for some fixed m value we have

$$e^{\alpha\frac{\partial}{\partial\theta}}(\theta^m) = \sum_{n=0}^{\infty}\frac{\alpha^n}{n!}\frac{\partial^n}{\partial\theta^n}\theta^m = \sum_{n=0}^{m}\frac{\alpha^n}{n!}n!\binom{m}{n}\theta^{m-n} = (\theta+\alpha)^m \tag{5.12}$$

due to the presence of the binomial formula. Hence we may write

$$\rho(R^{\hat{n}}_\alpha)(P) = \sum_{m}^{\infty} Q_m(r,h)e^{\alpha\frac{\partial}{\partial\theta}}\theta^m = \sum_{m=0}^{\infty}Q_m(r,h)(\theta+\alpha)^m = P(h,r,\theta+\alpha)\,. \tag{5.13}$$

Now we may set $\alpha=2\pi$ and observe our definition in (5.3) which shows that

$$P(r,h,\theta+2\pi) = P(r,h,\theta) \tag{5.14}$$

since we just have simple $\sin\theta$ and $\cos\theta$ terms, and so applying this to (5.13) we finally see that

$$\rho(R^{\hat{n}}_{2\pi})(P) = P(h,r,\theta+2\pi) = P(h,r,\theta) \tag{5.15}$$

and so as an operator on $\mathcal{P}_k$ we have $\rho(R^{\hat{n}}_{2\pi}) = 1$. □

Q6. ***E pur si muove***

Recall the set of harmonic polynomials on the sphere defined as

$$\mathcal{H}_k(S^2) = \{ P|_{S^2} \in \mathrm{Cts}(S^2, \mathbb{C}) \mid P \in \mathcal{P}_k(3) \quad \text{and} \quad \Delta_{\mathbb{R}^3} P = 0 \} \,. \tag{6.1}$$

In exercise L4-7 we saw that the map

$$\begin{aligned} \sigma : \mathrm{SO}(3)^{\mathrm{op}} &\longrightarrow \mathrm{Aut}_{\mathbb{C}}(\mathcal{H}_k(S^2)) \\ \sigma(R^{\hat{n}}_\alpha) &= \rho(R^{\hat{n}}_\alpha)\Big|_{\mathcal{H}_k(S^2)} \end{aligned} \tag{6.2}$$

is well defined, that is, if $P \in \mathcal{H}_k(S^2)$ then $[t_2 \frac{\partial}{\partial t_3} - t_3 \frac{\partial}{\partial t_2}](P) \in \mathcal{H}_k(S^2)$ also. Thus we may hope to be able to write down an expression for $\sigma(R^{\hat{n}}_\alpha)$ in terms of general spherical coordinates.

We begin by calculating $\sigma(R^x_\alpha)$, $\sigma(R^y_\alpha)$ and $\sigma(R^z_\alpha)$. In all cases we will make use of ∂_x, ∂_y and ∂_z so we will calculate these generically first. Using the same spherical parameterisation as in (3.4) but this time without r dependence since $\sigma(R^{\hat{n}}_\alpha)$ only acts on polynomials on S^2, hence $r = 1$, we may write

$$\theta = \arctan \frac{\sqrt{x^2+y^2}}{z} \,, \quad \psi = \arctan \frac{y}{x} \quad \text{for } x^2+y^2+z^2 = 1 \,, \;\; \text{so } \theta \in [0,\pi] \text{ and } \psi \in [0, 2\pi) \,. \tag{6.3}$$

Thus by the chain rule we have

$$\begin{aligned} \frac{\partial}{\partial x} = \frac{\partial}{\partial \theta}\frac{\partial \theta}{\partial x} + \frac{\partial}{\partial \psi}\frac{\partial \psi}{\partial x} &= \frac{x}{z\sqrt{x^2+y^2}} \frac{z^2}{x^2+y^2+z^2} \frac{\partial}{\partial \theta} - \frac{y}{x^2+y^2}\frac{\partial}{\partial \psi} \\ &= \cos\theta \cos\psi \frac{\partial}{\partial \theta} - \frac{\sin\psi}{\sin\theta}\frac{\partial}{\partial \psi} \,, \end{aligned} \tag{6.4}$$

and then we have

$$\begin{aligned} \frac{\partial}{\partial y} = \frac{\partial}{\partial \theta}\frac{\partial \theta}{\partial y} + \frac{\partial}{\partial \psi}\frac{\partial \psi}{\partial y} &= \frac{y}{z\sqrt{x^2+y^2}} \frac{z^2}{x^2+y^2+z^2} \frac{\partial}{\partial \theta} + \frac{x}{x^2+y^2}\frac{\partial}{\partial \psi} \\ &= \cos\theta \sin\psi \frac{\partial}{\partial \theta} + \frac{\cos\psi}{\sin\theta}\frac{\partial}{\partial \psi} \,, \end{aligned} \tag{6.5}$$

and then finally we have

$$\frac{\partial}{\partial z} = \frac{\partial}{\partial \theta}\frac{\partial \theta}{\partial z} + \frac{\partial}{\partial \psi}\frac{\partial \psi}{\partial z} = -\frac{\sqrt{x^2+y^2}}{z^2}\frac{z^2}{x^2+y^2+z^2}\frac{\partial}{\partial \theta} = -\sin\theta \frac{\partial}{\partial \theta} \,. \tag{6.6}$$

In the first case where we have $t_1 = x$, $t_2 = y$ and $t_3 = z$ for $\sigma(R^x_\alpha)$ our differential operator becomes

$$y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y} = -\sin^2\theta \sin\psi \frac{\partial}{\partial \theta} - \cos^2\theta \sin\psi \frac{\partial}{\partial \theta} - \frac{\cos\theta\cos\psi}{\sin\theta}\frac{\partial}{\partial \psi} \tag{6.7}$$

and so in simplifying this we have

$$\sigma(R^x_\alpha) = \exp\left(\alpha \left[-\sin\psi \frac{\partial}{\partial \theta} - \frac{\cos\theta\cos\psi}{\sin\theta}\frac{\partial}{\partial \psi} \right] \right) \,. \tag{6.8}$$

For the other cases we will appeal to the change of coordinates formula for a rotation about $\hat{n}$ from lectures, namely

$$\begin{pmatrix} t_1 \\ t_2 \\ t_3 \end{pmatrix} = T^{\hat{n}} x = [R^y_{\frac{\pi}{2}-\theta'} R^z_{-\psi'}]^S_S \begin{pmatrix} x \\ y \\ z \end{pmatrix} . \tag{6.9}$$

So, in the $\sigma(R^y_\alpha)$ case we desire a transformation such that $x \mapsto y$ but z is preserved. Using the diagram in L4-11.2, this corresponds to a transformation $\psi' = \frac{\pi}{2}$ and $\theta' = \frac{\pi}{2}$, hence giving

$$T^{\hat{n}} = R^z_{-\frac{\pi}{2}} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} , \quad \text{so} \quad (t_1, t_2, t_3) = (y, -x, z) . \tag{6.10}$$

Thus we have

$$\begin{aligned} t_2 \frac{\partial}{\partial t_3} - t_3 \frac{\partial}{\partial t_2} &= -x \frac{\partial}{\partial z} - z \frac{\partial}{\partial(-x)} = z \frac{\partial}{\partial x} - x \frac{\partial}{\partial z} \\ &= \cos\theta \left(\cos\theta \cos\psi \frac{\partial}{\partial\theta} - \frac{\sin\psi}{\sin\theta} \frac{\partial}{\partial\psi} \right) - \sin\theta\cos\psi \left(-\sin\theta \frac{\partial}{\partial\theta} \right) \\ &= \cos\psi \frac{\partial}{\partial\theta} - \frac{\cos\theta\sin\psi}{\sin\theta} \frac{\partial}{\partial\psi} . \end{aligned} \tag{6.11}$$

Finally we can look at $\sigma(R^z_\alpha)$ mapping $x \mapsto z$, so $\psi' = 0$ and $\theta' = 0$ which gives

$$T^{\hat{n}} = R^y_{\frac{\pi}{2}} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} , \quad \text{so} \quad (t_1, t_2, t_3) = (z, y, -x) , \tag{6.12}$$

which gives

$$\begin{aligned} t_2 \frac{\partial}{\partial t_3} - t_3 \frac{\partial}{\partial t_2} &= y \frac{\partial}{\partial(-x)} - (-x) \frac{\partial}{\partial y} = x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \\ &= \sin\theta\cos\psi \left(\cos\theta \sin\psi \frac{\partial}{\partial\theta} + \frac{\cos\psi}{\sin\theta} \frac{\partial}{\partial\psi} \right) - \sin\theta\sin\psi \left(\cos\theta\cos\psi \frac{\partial}{\partial\theta} - \frac{\sin\psi}{\sin\theta} \frac{\partial}{\partial\psi} \right) \\ &= \frac{\partial}{\partial\psi} . \end{aligned} \tag{6.13}$$

It is interesting to note that the formula for $\sigma(R^y_\alpha)$ appears to be the same as that for R^x_α after translating $\psi \mapsto \psi - \frac{\pi}{2}$, which would be consistent with our rotation matrices in (6.9). However, using such a relation appears to break down in the formula for $\sigma(R^z_\alpha)$. If one carries out the calculations, it can be seen that

$$R^z_{\psi'} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \cos(\psi+\psi')\sin\theta \\ \sin(\psi+\psi')\sin\theta \\ \cos\theta \end{pmatrix} ,$$

$$\text{but} \quad R^y_{\theta'-\frac{\pi}{2}} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \sin\theta'\sin\theta\cos\psi - \cos\theta'\cos\theta \\ \sin\theta\sin\psi \\ \cos\theta'\sin\theta\cos\psi + \sin\theta'\cos\theta \end{pmatrix} \neq \begin{pmatrix} -\cos(\theta+\theta')\cos\psi \\ \sin(\theta+\theta')\sin\psi \\ \sin(\theta+\theta')\cos\psi \end{pmatrix} .$$

All of this is to say, our hopes of being able to define spherical coordinates on cartesian coordinates first and then naturally build in the $\hat{n}$ rotation axis into that generic formula is fatally flawed by the incompatibility of the above calculation, despite working out in the $R^z_{\psi'}$ case. Sad!

Instead, suppose we have rotated our x-axis to be $\hat{n}$, so we are dealing with (t_1, t_2, t_3) coordinates. Then we may simply define new spherical coordinates

$$t_1 = \sin\theta\cos\psi\,, \quad t_2 = \sin\theta\sin\psi\,, \quad t_3 = \cos\theta \;\text{ where }\; \theta \in [0,\pi]\,, \quad \psi \in [0, 2\pi)\,. \tag{6.14}$$

In this case, one will just get the exact same formula as in (6.8). But this feels like cheating, no? It already presupposes that we are viewing the sphere from the perspective of our new coordinates (t_1, t_2, t_3).

Instead, suppose we want a formula that uses the standard xyz parametersation of the sphere but builds in the ψ' and θ' from $\hat{n}$. Then we may use our formula in (6.9) to write

$$\begin{aligned} t_1 &= \sin\theta' x + \sin\theta' y + \cos\theta' z \\ t_2 &= -\sin\psi' x + \cos\psi' y \\ t_3 &= -\cos\theta' x - \cos\theta'\sin\psi' y + \sin\theta' z\,. \end{aligned} \tag{6.15}$$

Using the invertibility and orthogonality of the rotation matrices then gives

$$\begin{aligned} \frac{\partial}{\partial t_1} &= \sin\theta'\cos\psi'\frac{\partial}{\partial x} + \sin\theta'\sin\psi'\frac{\partial}{\partial y} + \cos\theta'\frac{\partial}{\partial z} \\ \frac{\partial}{\partial t_2} &= -\sin\psi'\frac{\partial}{\partial x} + \cos\psi'\frac{\partial}{\partial y} \\ \frac{\partial}{\partial t_3} &= -\cos\theta'\cos\psi'\frac{\partial}{\partial x} - \cos\theta'\sin\psi'\frac{\partial}{\partial y} + \sin\theta'\frac{\partial}{\partial z}\,. \end{aligned} \tag{6.16}$$

Because I have already typed out enough lengthy chain rule questions in this assignment I am simply going to state the next equation without showing all of the simplification behind the scenes, left as an exercise to verify its legitimacy:

$$\begin{aligned} t_2\frac{\partial}{\partial t_3} - t_3\frac{\partial}{\partial t_2} &= \cos\theta'\left[x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right] + \sin\theta'\cos\psi'\left[y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right] + \sin\theta'\sin\psi'\left[z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right] \\ &= \left[\sin\theta'\sin\psi'\cos\psi - \sin\theta'\cos\psi'\sin\psi\right]\frac{\partial}{\partial\theta} \\ &\quad + \left[\cos\theta' - \frac{\sin\theta'\cos\psi'\cos\theta\cos\psi}{\sin\theta} - \frac{\sin\theta'\sin\psi'\cos\theta\sin\psi}{\sin\theta}\right]\frac{\partial}{\partial\psi} \\ &=: L(\theta, \psi, \partial_\theta, \partial_\psi\,;\; \theta', \psi')\,. \end{aligned} \tag{6.17}$$

We note that it *looks* like there is the potential application of trigonometric addition formulas, but as far as the eye can see they are only in our imagination. But! In using the (ψ', θ') values for each of $\sigma(R^x_\alpha)$, $\sigma(R^y_\alpha)$ and $\sigma(R^z_\alpha)$ as above, this formula perfectly marries up with our previous calculations which can be checked easily. Thus we finally have as an operator on $\mathcal{H}_k(S^2)$

$$\sigma(R^{\hat{n}}_\alpha) = \exp\left(\alpha L(\theta, \psi, \partial_\theta, \partial_\psi\,;\; \theta', \psi')\right)\,. \tag{6.18}$$

We note that we never want to see the chain rule ever again, nor write a partial derivative on LaTeX ever again. So. Much. Pain. □