Functional Analysis Assignment 4

Liam Carroll - 830916

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Q1. Minkowski functional

Let C be an absorbing subset of a vector space V, that is, for all $x \in V$ there is a non-negative α such that $x \in \alpha C$. Further, assume that if $x \in C$ and $0 \leq \tau \leq 1$, then $\tau x \in C$. Let $\rho : V \to \mathbb{R}^+$ be the Minkowski functional for C defined by

$$\rho(x) = \inf\{t \geq 0;\ x \in tC\}\,. \tag{1.1}$$

For $s \geq 0$, $\rho(sx) = s\rho(x)$.

Let $s \geq 0$ and $x \in V$ be arbitrary. We know from basic properties of the infimum that for $\lambda \geq 0$ and any set A, $\lambda \inf(A) = \inf(\lambda A)$ (i.e. set $M = \inf(\lambda A)$, then $M \leq \lambda x$ for all $x \in A$, so $M/\lambda \leq x$ for all $x \in A$, so $(1/\lambda)\inf(\lambda A) = \inf(A)$). Hence we can calculate

$$\begin{aligned} s\rho(x) &= s \inf\big(\{t \geq 0;\ x \in tC\}\big) \\ &= \inf\big(s\{t \geq 0;\ x \in tC\}\big) \\ &= \inf\big(\{st \geq 0;\ x \in tC\}\big) \\ &= \inf\left(\left\{st \geq 0;\ x \in \frac{(st)}{s}C\right\}\right) \\ &= \inf\big(\{st \geq 0;\ sx \in (st)C\}\big) \\ &= \inf\big(\{t' \geq 0;\ sx \in t'C\}\big) = \rho(sx)\,. \end{aligned} \tag{1.2}$$

$\{x : \rho(x) < 1\} \subset C \subset \{x : \rho(x) \leq 1\}$

First, we quickly prove that if $k < k'$ are two positive reals, then $kC \subset k'C$. Let $x \in kC$, so $x = kc$ for some $c \in C$. Then we have

$$\frac{x}{k'} = \underbrace{(k/k')}_{<1} c \in C \tag{1.3}$$

due to the scaling property hypothesised on C. Therefore, $x \in k'C$ so $kC \subset k'C$.

Let $x \in \{x : \rho(x) < 1\} \subset V$. Then $\rho(x) < 1$, so there exists a $t < 1$ such that $x \in tC$. Then choose $t' = 1$ and apply the above lemma to see that $x \in 1C = C$, showing the desired inclusion.

Now let $x \in C$, so $x \in 1C$ in particular. Then the set $\{t \geq 0 : x \in tC\}$ contains $t = 1$, meaning $\inf\{t \geq 0 : x \in tC\} \leq 1$, hence $\rho(x) \leq 1$ showing the desired inclusion.

If C is convex, then $\rho(x + y) \leq \rho(x) + \rho(y)$

Let C have all the properties as above, but also assume that it is convex, so if $x, y \in C$ and $0 \leq \tau \leq 1$ then $\tau x + (1 - \tau)y \in C$. We know that since $\rho(x)$ is an infimum, then for all $\varepsilon > 0$ there is t' such that $\rho(x) \leq t' < \rho(x) + \varepsilon$. Using this, we can set $t_x = \rho(x) + \varepsilon$ and $t_y = \rho(y) + \varepsilon$. Then clearly $\rho(x) + \varepsilon > \rho(x)$. By definition, we must have $x \in \rho(x)C$. Hence by our previous tiny lemma we have

$$x \in t_x C \qquad \text{and} \qquad y \in t_y C\,. \tag{1.4}$$

Then if we take the convex combination of the two quantities $x/t_x \in C$ and $y/t_y \in C$, we see

$$\frac{t_x}{t_x + t_y}\frac{x}{t_x} + \frac{t_y}{t_x + t_y}\frac{y}{t_y} = \frac{x + y}{t_x + t_y} \in C\,, \tag{1.5}$$

which means that

$$(x + y) \in (\rho(x) + \rho(y) + 2\varepsilon)C\,. \tag{1.6}$$

Therefore, the infimum is at worst $\rho(x) + \rho(y)$ (the 2ε vanishes when taking this inf), hence showing that

$$\rho(x + y) = \inf\big(\{t \geq 0;\; x + y \in tC\}\big) \leq \rho(x) + \rho(y)\,. \tag{1.7}$$

If C is circled then $\rho(\lambda x) = |\lambda|\rho(x)$

Assume C is circled - that is, if $x \in C$ and $\lambda \in \mathbb{C}$ satisifes $|\lambda| = 1$, then $\lambda x \in C$ - this is equivalent to saying that if $\lambda = e^{i\theta}$ then $e^{i\theta}x \in C$. In other words, this tells us that $e^{i\theta}C = C$ for any $\theta \in [0, 2\pi)$.

We now analyse $\rho(\lambda x)$ for the above hypothesis. Let $\lambda \in \mathbb{C}$ be arbitrary, so we can write $\lambda = |\lambda|e^{i\theta}$. Then

$$\begin{aligned}
\rho(\lambda x) &= \inf(\{t \geq 0\;;\; \lambda x \in tC\}) \\
&= \inf(\{t \geq 0\;;\; |\lambda|e^{i\theta}x \in tC\}) \\
&= \inf(\{t \geq 0\;;\; (|\lambda|/t)x \in e^{-i\theta}C\}) \\
&= \inf(\{t \geq 0\;;\; (|\lambda|/t)x \in C\}) \\
&= |\lambda|\rho(x)\,.
\end{aligned}$$

In the last line we used the property derived in part a).

□

In particular, all of these properties show that if C is convex and circled, then the Minkowski functional $\rho(x)$ is a well defined semi-norm.

Q2. Fréchet Space metric is indeed a metric

Let $\rho_j : X \to [0, \infty)$, $j \in \mathbb{N}$ be a countable family of seminorms on a vector space X that separates points - that is, for all $x \in X\backslash\{0\}$, there is a $k \in \mathbb{N}$ such that $\rho_k(x) \neq 0$. We note also that a seminorm ρ_j is a norm that is not positive definite, so it obeys the triangle inequality and absolute homogeneity. We can then define the Fréchet metric,

$$d(x, y) = \sum_{j=1}^{\infty} 2^{-j} \frac{\rho_j(x-y)}{1+\rho_j(x-y)} \,. \tag{2.1}$$

It is first worth noting that by the ratio test (where the fraction is clearly less than 1 due to the positivity of the semi-norms), this series is indeed well defined in the sense that is convergent.

It is obvious with the absolute homogeneity that we have symmetry. It is also clear that $d(x, y) \geq 0$ since the seminorms have this same property. Further, if $x = y$ then $\rho_j(x - y) = \rho_j(0) = |0|\rho_j(x) = 0$, so $d(x, y) = 0$ as well.

If $d(x, y) = 0$, then since it is a sum of non-negative terms, we must have

$$\text{for all } \; j \in \mathbb{N}\,, \quad \rho_j(x-y) = 0\,. \tag{2.2}$$

But since the family of semi-norms separates points, the only element of X that satisfies this condition is 0, hence $x - y = 0$.

Clearly though for the triangle inequality we have some work to do. The sticking point will clearly be separating the fraction, so we first consider the function $f : \mathbb{R}^+ \to \mathbb{R}^+$ defined by

$$f(a) = \frac{a}{1+a} = 1 - \frac{1}{1+a}\,. \tag{2.3}$$

We first see that $f(a)$ is increasing since $1/(1 + a)$ is decreasing. We then want to show the triangle inequality, $f(a + b) \leq f(a) + f(b)$ for $a, b \in \mathbb{R}^+$. We observe that

$$\frac{f(a)}{a} = \frac{1}{1+a} \geq \frac{1}{1+a+b} = \frac{f(a+b)}{a+b}\,, \quad \text{and similarly } \; \frac{f(b)}{b} \geq \frac{f(a+b)}{a+b}\,, \tag{2.4}$$

which respectively gives us

$$(a+b)f(a) \geq af(a+b) \qquad \text{and} \qquad (a+b)f(b) \geq bf(a+b)\,,$$

and so combining the two inequalities gives

$$(a+b)(f(a)+f(b)) \geq (a+b)f(a+b)\,. \tag{2.5}$$

Putting this inequality and the fact that f is increasing together, we have for $a, b \in \mathbb{R}^+$,

$$f(a) \leq f(a+b) \leq f(a) + f(b)\,. \tag{2.6}$$

Clearly we will then use $a = \rho_j(x-y)$ and $b = \rho_j(y-z)$ to establish the triangle inequality for the metric.

We calculate

$$\begin{aligned} d(x,z) &= \sum_{j=1}^{\infty} 2^{-j} \frac{\rho_j(x-z)}{1+\rho_j(x-z)} \\ &\leq \sum_{j=1}^{\infty} 2^{-j} \frac{\rho_j(x-y)+\rho_j(y-z)}{1+\rho_j(x-y)+\rho_j(y-z)} \\ &\leq \sum_{j=1}^{\infty} 2^{-j} \left(\frac{\rho_j(x-y)}{1+\rho_j(x-y)} + \frac{\rho_j(y-z)}{1+\rho_j(y-z)} \right) \\ &= \sum_{j=1}^{\infty} 2^{-j} \frac{\rho_j(x-y)}{1+\rho_j(x-y)} + \sum_{j=1}^{\infty} 2^{-j} \frac{\rho_j(y-z)}{1+\rho_j(y-z)} \\ &= d(x,y) + d(y,z)\,. \end{aligned} \tag{2.7}$$

In the second line we used the triangle inequality of the seminorms and fact that f was increasing. In the third line we used the triangle inequality from (2.6). Therefore the metric d obeys the triangle inequality and all other metric properties, hence d is a metric, thus giving us the fact that a locally convex vector space whose topology is generated by a countable family of seminorms that separates points is metrizable.

□

Q4. Compact operators in different topologies

Part a)

We first construct an example of a sequence of compact operators $K_n : \ell^2 \to \ell^2$ such that $K_n \to Id$ in the strong operator topology on $\mathcal{L}(\ell^2)$. We first note that the obvious choice may be to simply take $K_n = Id$ - however, the identity is not actually a compact operator. We know that the unit ball in the Banach space ℓ^2 is not compact, i.e. the bounded sequence $(e_i)_{i=1}^{\infty}$ of unit vectors $e_i \in \ell^2$, have no convergent subsequence. Therefore $(Id\ e_i)_{i=1}^{\infty} = (e_i)_{i=1}^{\infty}$ also doesn't have a convergent subsequence, hence meaning the identity cannot be a compact operator.

Instead, we know from lectures that finite-rank operators are compact. We can quickly confirm this. Take a bounded sequence $(x^{(j)})_{j=1}^{\infty}$ of elements $x^{(j)} \in \ell^2$ and define K_n to be the projection of $x^{(j)}$ on to $\mathbb{C}^n$ - that is, given

$$x^{(j)} = (x_1^{(j)}, x_2^{(j)}, x_3^{(j)}, x_4^{(j)}, \dots)\,, \tag{4.1}$$

we define for $n \in \mathbb{N}$

$$K_n x^{(j)} = (x_1^{(j)}, \dots, x_n^{(j)}, 0, 0, \dots)\,. \tag{4.2}$$

Then the sequence $(K_n x^{(j)})_{j=1}^{\infty}$, which reads as

$$\begin{aligned} K_n x^{(1)} &= (x_1^{(1)}, \dots, x_n^{(1)}, 0, 0, \dots) \\ K_n x^{(2)} &= (x_1^{(2)}, \dots, x_n^{(2)}, 0, 0, \dots) \\ &\vdots \end{aligned} \tag{4.3}$$

is bounded in $\mathbb{C}^n$ since by hypothesis $(x^{(j)})_{j=1}^{\infty}$ is bounded. Then, by the Bolzano Weierstrass Theorem, any bounded sequence in $\mathbb{C}^n \cong \mathbb{R}^{2n}$ admits a convergent subsequence. Therefore we must have $(K_n x^{(j)})_{j=1}^{\infty}$ having a convergent subsequence, therefore showing that the operator K_n is a compact operator.

Clearly then we have found a sequence of compact operators $(K_n)_{n=1}^{\infty}$ that converge to the identity in the strong operator topology since we have, for $x \in \ell^2$,

$$\lim_{n\to\infty} \|K_n x - Id x\|_2 = \lim_{n\to\infty} \sum_{j=n+1}^{\infty} |x_j|^2 = 0\,, \tag{4.4}$$

since it is the tail of a necessarily convergent sequence in ℓ^2. Therefore, $K_n \to Id$ in the strong operator topology (since we know $K_n \to K$ in SOT if and only if for all $x \in X$ we have $K_n x \to Kx$ in X). $\square$

Part b)

Let X and Y be Banach spaces. From lectures we know that the space of compact operators $\mathcal{K}(X, Y)$ is a closed subset of the space of bounded operators $\mathcal{L}(X, Y)$ in the norm topology, that is, the topology induced by the operator norm. That is, if $K_n \in \mathcal{K}(X, Y)$ is a sequence of compact operators for $n \in \mathbb{N}$, then if $K_n \to K$ in the norm topology, then K is also compact (since a closed subset must contain its limit points). Clearly then we can take the contrapositive of the this statement - if K is not compact, then the sequence of compact operators K_n *cannot* converge in the norm topology to K.

Now let $(K_n)_{n=1}^{\infty}$ be any sequence of compact operators in $\mathcal{L}(\ell^2)$ which converge in the strong operator topology to Id. We argued in part a) that the identity operator Id is not compact. Therefore by the above argument, we cannot have $K_n \to Id$ in the norm topology. $\square$

N.B. we hypothesised that $K_n \to Id$ in the strong norm topology simply because we know that for $\mathcal{L}(X, Y)$ we have

$$\text{Weak operator topology} \subseteq \text{Strong operator topology} \subseteq \text{Norm topology}\,,$$

meaning it would be redundant to talk about convergence to the identity in the norm topology if it didn't converge in the strong operator topology in the first place.

Q5. Spectral radius of Volterra operator

Let X be a Banach space. For any $T \in \mathcal{L}(X)$, the spectral radius is defined as

$$r(T) := \sup\{|\lambda| : \lambda \in \sigma(T)\}\,, \tag{5.1}$$

$$\text{where} \quad \sigma(T) = \{\lambda \in \mathbb{C} : \lambda I - T \in \mathcal{L}(X) \text{ is not invertible}\}\,. \tag{5.2}$$

Further, we know from lectures that under these hypothesis we also have

$$r(T) = \lim_{n\to\infty} \|T^n\|_{\mathcal{L}(X)}^{1/n}\,. \tag{5.3}$$

We consider the spectral radius of the Volterra integral operator on $X = (C^0[0,1], \|.\|_0)$ defined as

$$(Tf)(x) = \int_0^x f(y)dy\,. \tag{5.4}$$

We can start by calculating

$$\|T\|_{\mathcal{L}(X)} = \sup\{\|Tf\|_0 : \|f\|_0 = 1\}\,. \tag{5.5}$$

Let $f \in C^0[0,1]$ be such that $\|f\|_0 = 1$, then we have

$$\|Tf\|_0 = \sup_{x\in[0,1]} \left|\int_0^x f(y)dy\right| \leq \sup_{x\in[0,1]} \int_0^x |f(y)|dy \leq \sup_{x\in[0,1]} \int_0^x 1dy = 1\,. \tag{5.6}$$

But then by noting that for $f(x) = 1$ we have

$$\|Tf\|_0 = \sup_{x\in[0,1]} \left|\int_0^x 1dy\right| = \sup_{x\in[0,1]} |x| = 1\,, \tag{5.7}$$

so clearly we must have $\|T\|_{\mathcal{L}(X)} = 1$. We then seek to calculate $\|T^n\|_{\mathcal{L}(X)}$ where we have

$$(T^n f)(x) = \int_0^x \int_0^{y_1} \int_0^{y_2} \cdots \int_0^{y_{n-1}} f(y_n)dy_n \ldots dy_2\, dy_1\,. \tag{5.8}$$

We then appeal to Cauchy's formula for repeated integration (which can be proven with a very simple induction argument) which allows us to equivalently write

$$(T^n f)(x) = \frac{1}{(n-1)!}\int_0^x (x-y)^{n-1} f(y)dy\,. \tag{5.9}$$

Then performing the same calculation as above for $\|f\|_0 = 1$ we have

$$\begin{aligned}
\|Tf\|_0 &= \sup_{x\in[0,1]} \left|\frac{1}{(n-1)!}\int_0^x (x-y)^{n-1} f(y)dy\right| \\
&\leq \sup_{x\in[0,1]} \frac{1}{(n-1)!}\int_0^x (x-y)^{n-1}dy \\
&= \sup_{x\in[0,1]} \frac{1}{(n-1)!}\left[-\frac{1}{n}(x-y)^n\right]_0^x \\
&= \sup_{x\in[0,1]} \frac{1}{n!}x^n = \frac{1}{n!}\,.
\end{aligned} \tag{5.10}$$

In the second line we used the fact that $(x-y)^{n-1} \geq 0$ for $0 \leq y \leq x$, hence we could drop the absolute value. Again noting that we could simply choose $f(x) = 1$ as before, this gives us

$$\|T^n\|_{\mathcal{L}(X)} = \frac{1}{n!} \,. \tag{5.11}$$

We then note that we have

$$\lim_{n\to\infty} (n!)^{1/n} = \infty \quad \text{since } e^x \geq \frac{x^n}{n!} \,, \quad \text{so } \ (n!)^{1/n} \geq \frac{n}{e} \to \infty \,. \tag{5.12}$$

Therefore, since we are taking the reciprocal of this limit, we have

$$r(T) = \lim_{n\to\infty} \|T^n\|^{1/n}_{\mathcal{L}(X)} = \lim_{n\to\infty} \left(\frac{1}{n!}\right)^{1/n} = 0 \,. \tag{5.13}$$

Then since we know that the spectrum is a non-empty subset (due to the analyticity of the resolvent function), we have

$$\sigma(T) = \{0\} \,. \tag{5.14}$$

□