Functional Analysis Assignment 2

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Q1. Bi-infinite sequences and Sobolev space

Define $x = \{x_n\}_{n=-\infty}^{\infty}$ to be a bi-infinite sequence with $x_n \in \mathbb{C}$, indexed by $\mathbb{Z}$. Let $s \in \mathbb{R}$ be given. We define a norm on x as

$$\|x\|_{h_s} := \left(\sum_{n=-\infty}^{\infty} (1+n^2)^s |x_n|^2 \right)^{1/2} , \tag{1.1}$$

and the L^2-based Sobolev space of order s as

$$h_s = \{x = \{x_n\}_{n=-\infty}^{\infty} : \|x\|_{h_s} < \infty\} . \tag{1.2}$$

Part a)

We first show that h_s is a normed linear space.

(i) $\|x\|_{h_s} = 0 \iff x = 0$
Suppose for $x \in h_s$ we have $\|x\|_{h_s} = 0$. Then, taking squares we have

$$\sum_{n=-\infty}^{\infty} (1+n^2)^s |x_n|^2 = 0 ,$$

but clearly all terms $(1+n^2)^s|x_n|^2 \geq 0$, and $(1+n^2)^s \neq 0$ for any value of $n \in \mathbb{Z}$ or $s \in \mathbb{R}$, so we must have $|x_n|^2 = 0$ for all n, so $x = 0$. The other direction is obvious.

(ii) $\|\alpha x\|_{h_s} = |\alpha| \|x\|_{h_s}$ for all $\alpha \in \mathbb{C}$
We calculate for $\alpha \in \mathbb{C}$ and $x \in h_s$

$$\|\alpha x\|_{h_s} = \left(\sum_{n=-\infty}^{\infty} (1+n^2)^s |\alpha x_n|^2 \right)^{1/2} = \left(|\alpha|^2 \sum_{n=-\infty}^{\infty} (1+n^2)^s |x_n|^2 \right)^{1/2} = |\alpha| \|x\|_{h_s} . \tag{1.3}$$

(iii) $\|x+y\|_{h_s} \leq \|x\|_{h_s} + \|y\|_{h_s}$ for all $x, y \in h_s$

This bi-infinte business is clearly frustrating to work with - we like sequences in $\mathbb{N}$, not $\mathbb{Z}$. So lets work with $\mathbb{N}$ instead.

We can formulate the natural bijection between $\mathbb{Z}$ and $\mathbb{N}$ by sending the positive integers to the even naturals, and the negative integers to the odd naturals. That is, consider $f : \mathbb{Z} \to \mathbb{N}$ and $f^{-1} : \mathbb{N} \to \mathbb{Z}$ defined by

$$f(n) = \begin{cases} 2n & n \geq 0 \\ -2n-1 & n < 0 \end{cases}, \quad \text{and} \quad f^{-1}(k) = (-1)^k \left\lceil \frac{k}{2} \right\rceil . \tag{1.4}$$

It is clear f does indeed define a bijection. Then in setting $k = f(n)$ and $n = f^{-1}(k)$ we can rewrite out sum of interest for $x, y \in h_s$

$$\begin{aligned} \|x+y\|_{h_s}^2 &= \sum_{n=-\infty}^{\infty} (1+n^2)^s |x_n + y_n|^2 \\ &= \sum_{k=0}^{\infty} \left(1 + \left((-1)^k \left\lceil \frac{k}{2} \right\rceil\right)^2\right)^s |x_{f^{-1}(k)} + y_{f^{-1}(k)}|^2 \\ &= \sum_{k=0}^{\infty} \left(1 + \left\lceil \frac{k}{2} \right\rceil^2\right)^s |x_{f^{-1}(k)} + y_{f^{-1}(k)}|^2 , \end{aligned} \tag{1.5}$$

where we are permitted to rearrange these terms since $x, y \in h_s$ gives us that $x+y \in h_s$ with elementary real analysis arguments, meaning $x+y$ is absolutely convergent. We can then assume the Minkowski inequality,

$$\left(\sum_{k=0}^{\infty} |x_k + y_k|^p\right)^{1/p} \leq \left(\sum_{k=0}^{\infty} |x_k|^p\right)^{1/p} + \left(\sum_{k=0}^{\infty} |y_k|^p\right)^{1/p} , \tag{1.6}$$

to conclude that

$$\begin{aligned} \|x+y\|_{h_s} &= \left(\sum_{n=-\infty}^{\infty} (1+n^2)^s |x_n + y_n|^2\right)^{1/2} \\ &= \left(\sum_{k=0}^{\infty} \left(1 + \lceil k/2 \rceil^2\right)^s |x_{f^{-1}(k)} + y_{f^{-1}(k)}|^2\right)^{1/2} \\ &\leq \left(\sum_{k=0}^{\infty} \left|\left(1 + \lceil k/2 \rceil^2\right)^{s/2} x_{f^{-1}(k)}\right|^2\right)^{1/2} + \left(\sum_{k=0}^{\infty} \left|\left(1 + \lceil k/2 \rceil^2\right)^{s/2} y_{f^{-1}(k)}\right|^2\right)^{1/2} \\ &= \left(\sum_{n=-\infty}^{\infty} (1+n^2)^s |x_n|^2\right)^{1/2} + \left(\sum_{n=-\infty}^{\infty} (1+n^2)^s |y_n|^2\right)^{1/2} \\ &= \|x\|_{h_s} + \|y\|_{h_s} , \end{aligned} \tag{1.7}$$

which proves the triangle inequality as desired.

Thus we conclude h_s is a normed linear space. $\square$

Part b)

The natural inner product $\langle\ .\ ,\ .\ \rangle : h_s \times h_s \to \mathbb{C}$ to define that induces the $\|.\|_{h_s}$ norm is, for $x, y \in h_s$,

$$\langle x, y\rangle = \sum_{n=-\infty}^{\infty} (1+n^2)^s\, \overline{x_n}\, y_n\,. \tag{1.8}$$

This clearly induces the norm since

$$\langle x, x\rangle = \sum_{n=-\infty}^{\infty} (1+n^2)^s\, \overline{x_n} x_n = \sum_{n=-\infty}^{\infty} (1+n^2)^s\, |x_n|^2 = \|x\|_{h_s}^2\,. \tag{1.9}$$

We can then prove this is a well defined inner product:

(i) $\langle x+y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$ for all $x, y, z \in h_s$

We have

$$\begin{aligned}\langle x+y, z\rangle &= \sum_{n=-\infty}^{\infty} (1+n^2)^s (\overline{x_n + y_n}) z_n \\ &= \sum_{n=-\infty}^{\infty} (1+n^2)^s (\overline{x_n} + \overline{y_n}) z_n \\ &= \sum_{n=-\infty}^{\infty} (1+n^2)^s \overline{x_n} z_n + \sum_{n=-\infty}^{\infty} (1+n^2)^s \overline{y_n} z_n \\ &= \langle x, z\rangle + \langle y, z\rangle\,. \end{aligned} \tag{1.10}$$

(ii) $\langle x, x\rangle \geq 0$ and $\langle x, x\rangle = 0 \implies x = 0$ for all $x \in h_s$

This is clear due to our identification of the norm in (1.9), hence we just use these properties derived in part a.

(iii) $\langle x, \alpha y\rangle = \alpha\langle x, y\rangle$ for all $x, y \in h_s$ and $\alpha \in \mathbb{C}$

We calculate

$$\langle x, \alpha y\rangle = \sum_{n=-\infty}^{\infty} (1+n^2)^s\, \overline{x_n}\, \alpha y_n = \alpha \sum_{n=-\infty}^{\infty} (1+n^2)^s\, \overline{x_n}\, y_n = \alpha\langle x, y\rangle\,. \tag{1.11}$$

(iv) $\langle x, y\rangle = \overline{\langle y, x\rangle}$ for all $x, y \in h_s$

Noting that $s \in \mathbb{R}$, we have

$$\overline{\langle y, x\rangle} = \overline{\sum_{n=-\infty}^{\infty} (1+n^2)^s\, \overline{y_n}\, x_n} = \sum_{n=-\infty}^{\infty} \overline{(1+n^2)^s\, \overline{y_n}\, x_n} = \sum_{n=-\infty}^{\infty} (1+n^2)^s\, \overline{x_n}\, y_n = \langle x, y\rangle\,. \tag{1.12}$$

Therefore our defined inner product is well defined, so $(h_s, \langle\ ,\ \rangle)$ is an inner product space. □

Part c)

We now want to show that h_s is complete in the $\|.\|_{h_s}$ norm. We can do this by identifying it with ℓ^2 space. We will appeal to our rewritten summation formula in (1.5) to write, for $x \in h_s$ and f as defined in (1.4),

$$\|x\|_{h_s} = \left(\sum_{n=-\infty}^{\infty} (1+n^2)^s |x_n|^2 \right)^{1/2} = \left(\sum_{k=0}^{\infty} \left| \left(1 + \lceil k/2 \rceil^2\right)^{s/2} x_{f^{-1}(k)} \right|^2 \right)^{1/2} . \quad (1.13)$$

Since $x \in h_s$ we know that $\|x\|_{h_s}$ is finite. This leads us to defining a map $g : h_s \to \ell^2$ and $g^{-1} : \ell^2 \to h_s$ with

$$h_s \ni x \mapsto g(x) = \left\{ \left(1 + \lceil k/2 \rceil^2\right)^{s/2} x_{f^{-1}(k)} \right\}_{k=0}^{\infty} , \quad (1.14)$$

$$\text{and} \quad \ell^2 \ni \tilde{x} \mapsto g^{-1}(\tilde{x}) = \left\{ \left(1+n^2\right)^{-s/2} \tilde{x}_{f(n)} \right\}_{n=-\infty}^{\infty} . \quad (1.15)$$

Just as a sanity check to make sure we don't get too bogged down in the notation here, we note that

$$(g^{-1} \circ g)(x) = \left\{ (1+n^2)^{-s/2} (1 + f^{-1}(f(n))^2)^{s/2} \, x_{f^{-1}(f(n))} \right\}_{n=-\infty}^{\infty} = \{x_n\}_{n=-\infty}^{\infty} , \quad (1.16)$$

and similarly for $(g \circ g^{-1})(\tilde{x})$. This map is well defined since, with the standard norm on ℓ^2, we have

$$\|x\|_{h_s} = \|g(x)\|_2 , \quad (1.17)$$

and since $\|x\|_{h_s}$ is finite, then $\|g(x)\|_2$ is clearly finite as well - i.e. for any $x \in h_s$ we have $g(x) \in \ell^2$. Noting the bijection arguments from the triangle inequality proof in part a), we can conclude that g is a bijection between h_s and ℓ^2. We also note that since g is essentially just a rearrangement of terms, using the standard definitions on sequence spaces, it is clear that g is linear.

In the first assignment, we proved that ℓ^p space is complete. Clearly then, if we take a Cauchy sequence of elemenets $X^{(j)} = \{x^{(j)}\}_{j=1}^{\infty}$ where $x^{(j)} = \{x_n^{(j)}\}_{n=1}^{\infty} \in h_s$, then we know that there exists an element $Y \in \ell^2$ such that

$$\lim_{j \to \infty} \|g(X^{(j)}) - Y\|_2 = 0 . \quad (1.18)$$

Using the linearity of g and the fact that $g^{-1}(Y) \in h_s$ is well defined, we have

$$0 = \lim_{j \to \infty} \|g(X^{(j)}) - Y\|_2 = \lim_{j \to \infty} \|g(X^{(j)} - g^{-1}(Y))\|_2 = \lim_{j \to \infty} \|X^{(j)} - g^{-1}(Y)\|_{h_s} , \quad (1.19)$$

which means we have found our candidate limit!

Therefore, we conclude that for a Cauchy sequence $X^{(n)}$ of sequences in h_s, there exists a $Y \in \ell^2$, so a $g^{-1}(Y) \in h_s$ such that $X^{(n)}$ converges to $g^{-1}(Y)$ under the $\|.\|_{h_s}$ norm. In other words, $(h_s, \|.\|_{h_s})$ is a Banach space. □

Q2. Separability of ℓ^p space (and friends)

We first prove that ℓ^p is separable - that is, there exists a subset $A \subset \ell^p$ such that A is both countable and dense in ℓ^p.

We first claim that the set of finite sequences S is dense in ℓ^p where we define

$$S := \left\{ x^{(n)} = \{x_i^{(n)}\}_{i=0}^{\infty} \;\middle|\; \begin{array}{l} \text{for all } i \in \mathbb{N}_0,\ x_i^{(n)} \in \mathbb{C}, \\ \text{if } i \le n,\ x_i^{(n)} = x_i, \\ \text{if } i > n,\ x_i^{(n)} = 0 \end{array} \right\}. \tag{2.1}$$

To prove $S \subset \ell^p$ is dense, we need to show that any $x \in \ell^p$ can be expressed as a limit of a sequence in S. Let $x \in \ell^p$ with $x = \{x_i\}_{i=0}^{\infty}$ be given and consider the sequence of elements $X = \{x^{(n)}\}_{n=0}^{\infty}$ where $x^{(n)} \in S$. Given that x is in ℓ^p we have that $\|x\|_p$ is finite which implies that x must converge to 0. Then

$$\lim_{n\to\infty} \|x - x^{(n)}\|_p = \lim_{n\to\infty} \left(\sum_{i=0}^{\infty} |x_i - x_i^{(n)}|^p \right)^{1/p} = \left(\lim_{n\to\infty} \sum_{i=n+1}^{\infty} |x_i|^p \right)^{1/p} = 0\,, \tag{2.2}$$

where the final equality is due to the fact that x converges to 0. Hence this shows that $S \subset \ell^p$ is dense. It remains to find a countably dense subset. Clearly, since we are working with $\mathbb{C} \cong \mathbb{R}^2$, we should investigate $\mathbb{Q}^2$ which is both countable and dense in $\mathbb{R}^2$. That is, for any $x_i^{(n)} \in \mathbb{C}$ we can find a sequence

$$z^{(n,i)} = \{z_j^{(n,i)}\}_{j=0}^{\infty} = \{a_j^{(n,i)}\}_{j=0}^{\infty} + \sqrt{-1}\{b_j^{(n,i)}\}_{j=0}^{\infty}\,, \quad \text{where} \quad a_j^{(n,i)},\, b_j^{(n,i)} \in \mathbb{Q}\,, \tag{2.3}$$

such that $x_i^{(n)}$ can be approximated by this sequence. That is, by the density of $\mathbb{Q}$ we have

$$\lim_{j\to\infty} \|x_i^{(n)} - z_j^{(n,i)}\|_{\mathbb{C}} \le \lim_{j\to\infty} \left(|\mathrm{Re}(x_i^{(n)}) - a_j^{(n,i)}| + |\mathrm{Im}(x_i^{(n)}) - b_j^{(n,i)}| \right) = 0\,. \tag{2.4}$$

This shows that every element $x_i^{(n)}$ of a sequence $x^{(n)} \in S$ can be well approximated by sequences $z^{(n,i)}$, that is, (2.4) tells us

$$\|x^{(n)} - z^{(n,i)}\|_p^p \le \sum_{i=0}^{\infty} |x_i^{(n)} - z^{(n,i)}|^p \to 0\,. \tag{2.5}$$

Putting all of this together, we see that if we define

$$A := \left\{ z^{(n)} = \{z^{(n,i)}\}_{i=0}^{\infty} \;\middle|\; \begin{array}{l} z^{(n)} \text{ is finite as in } S \text{ and,} \\ z^{(n,i)} \text{ can be approximated by the rational} \\ \quad \text{sequence } \{z_j^{(n,i)}\}_{j=0}^{\infty} \text{ for } z_j^{(n,i)} \in \mathbb{Q}^2 \end{array} \right\} \tag{2.6}$$

then we see that A is countable (sequences of $\mathbb{Q}^2$ which is countable), and most importantly $A \in \ell^p$ is dense since, using (2.2) and (2.4) we have for any $x \in \ell^p$, there exists a sequence $z^{(n)} = \{z^{(n,i)}\}_{i=0}^{\infty} \in A$ with

$$\|x - z^{(n,i)}\|_p \le \|x - z^{(n)}\|_p + \|z^{(n)} - z^{(n,i)}\|_p \to 0\,. \tag{2.7}$$

□

We then consider $(c_0, \|.\|_\infty)$, the space of sequences that converge to 0. But this can be done using an identical argument as above and merely replacing the p-norm with the sup-norm. Then finite sequences are still dense in c_0 since

$$\lim_{n\to\infty} \|x - x^(n)\|_\infty = \lim_{n\to\infty} \sup_{n\in\mathbb{N}} |x - x^{(n)}| = 0\,, \tag{2.8}$$

and we can still approximate each complex element by rationals that will also converge in the sup-norm. Thus the argument is the same so c_0 is also separable. □

To show that $(l^\infty, \|.\|_\infty)$ is not separable, we wish to show that every dense subset of l^∞ is uncountable. We start by constructing a set of open balls in ℓ^∞. Let $I \subset \mathbb{N}$ be an index set. We can construct a sequence $e_I \in \ell^\infty$ as

$$e_I = \{e_I^i\}_{i=0}^\infty \quad \text{where} \quad e_I^i = \begin{cases} 1 & \text{if } i \in I, \\ 0 & \text{if } i \notin I \end{cases}. \tag{2.9}$$

Consider then any other index set $J \neq I \subset \mathbb{N}$ and its corresponding sequence e_J. Then we have

$$\|e_I - e_J\|_\infty = \sup_{i\in\mathbb{N}} |e_I^i - e_J^i| = 1\,. \tag{2.10}$$

We can then construct an open set U of open *disjoint* balls surrounding the point e_I,

$$U := \{B(e_I, \varepsilon = 1/2)\,|\, I \subset \mathbb{N}\}\,, \tag{2.11}$$

where disjointness follows from (2.10) (i.e. the closest sequence e_J to e_I is at least 1 away). We then see that $\mathrm{Card}(U) = \mathrm{Card}(\mathcal{P}(\mathbb{N}))$ and by Cantor's theorem about the cardinality of power sets, this shows that U is uncountable.

Let $C \subset \ell^\infty$ be a dense subset. By the definition of density, any neighbourhood of a point $x \in \ell^\infty$ must contain a point $y \in C$. That is, every open ball $B \in U$ must contain a point $y \in C$, but since these balls are disjoint, these points y must be distinct. Therefore for any dense subset C we must have

$$\mathrm{Card}(\{y \in C : y \in B \in U\}) = \mathrm{Card}(U)\,, \tag{2.12}$$

but U is uncountable from above, which proves that every dense subset of ℓ^∞ is uncountable, which shows ℓ^∞ is *not* separable. □

Q3. Isomorphism of quotient space to orthogonal complement

Let X be a Hilbert space and let $M \subset X$ be a closed subspace. We want to prove that for the natural map,

$$\begin{aligned} \pi\colon X &\to X/M = \{x + M : x \in X\} \\ x &\mapsto [x] = x + M\,, \end{aligned} \tag{3.1}$$

the restriction $\pi|_{M^\perp} : M^\perp \to X/M$ is an isomorphism of $M^\perp$ and X/M - that is, it is a bijective isometry. We define

$$M^\perp = \{x \in X : \text{for all } m \in M, \langle m, x\rangle = 0\}\,, \tag{3.2}$$

$$\text{and} \quad \text{for } [x] \in X/M, \quad \|[x]\|_{X/M} = \inf_{m \in M} \|x + m\|_X\,, \tag{3.3}$$

where we note from lectures that the norm on X/M is well defined. We also know that for a closed subspace $M \subset X$ we can write $X = M \oplus M^\perp$. From now on for notational convenience, assume that π refers to the restricted map $\pi|_{M^\perp}$.

(i) π is surjective

Let $[y] \in X/M$ be given. We want to show that there exists $x \in M^\perp$ such that $\pi(x) = [y]$. We first note that for the trivial case $[y] = [0]$ we clearly have $0 \in M^\perp$ and $\pi(0) = [0]$ is satisfied, so assume that $[y] \neq [0]$. Take an element $y + m \in [y]$ for $m \in M$ and $y \in X \backslash M$. We know that for any $h \in X$ we can write $h = h_m + h_{M^\perp}$, which tells us that $y \in M^\perp$. Hence, for any $[y] \in X/M$ we are guaranteed to have $y \in M^\perp$ such that $\pi(y) = [y]$, so π is surjective.

(ii) π is injective

Let $x, y \in M^\perp$ be such that $\pi(x) = \pi(y)$. Then $[x] = [y]$, so $[x - y] = [0]$, so $x - y \in M$. But since $x, y \in M^\perp$, this gives us for all $m \in M$ that

$$\langle m, x\rangle = 0 \quad \text{and} \quad \langle m, y\rangle = 0, \quad \text{so} \quad \langle m, x\rangle - \langle m, y\rangle = \langle m, x - y\rangle = 0\,, \tag{3.4}$$

so we also have $x - y \in M^\perp$. But since $M \cap M^\perp = \{0\}$, this implies $x - y = 0$ so clearly $x = y$ and so π is injective.

(iii) π is an isometry

We want to show that π, which is clearly linear, is an isometry which is equivalent to being norm-preserving due to that linearity. That is, for all $x \in M^\perp \subset X$ we want to show $\|x\|_X = \|\pi(x)\|_{X/M}$. Then we calculate, using $\langle x, m\rangle = 0$ for all $m \in M$ and $\inf_{m \in M} \|m\|_X^2 = 0$ since $0 \in M$,

$$\begin{aligned} \|\pi(x)\|_{X/M}^2 = \inf_{m \in M} \|x + m\|_X^2 &= \inf_{m \in M} \langle x + m, x + m\rangle \\ &= \inf_{m \in M} \big(\langle x, x\rangle + 2\mathrm{Re}\langle x, m\rangle + \langle m, m\rangle\big) \\ &= \|x\|_X^2 + \inf_{m \in M} \|m\|_X^2 = \|x\|_X^2\,, \end{aligned}$$

which shows the desired equality, hence π is an isometry.

Therefore $\pi|_{M^\perp}$ induces the isomorphism $M^\perp \cong X/M$. $\square$

Q4. Duals and not so duals

We first want to show that $\ell_1^* \cong \ell_\infty$. Define

$$\ell_1^* = \left\{ L_x : \ell_1 \to \mathbb{C} \;\middle|\; \|L_x\|_{\ell_1^*} := \sup_{y \in \ell_1 \setminus \{0\}} \frac{\|L_x(y)\|_{\mathbb{C}}}{\|y\|_1} < \infty \right\}, \tag{4.1}$$

where we define the map

$$\begin{aligned} \Phi : \ell_\infty &\longrightarrow \ell_1^* \\ x = \{x_n\}_{n=1}^\infty &\longmapsto L_x : \ell_1 \longrightarrow \mathbb{C} \quad \text{where} \\ & \qquad L_x(y) = \sum_{n=1}^\infty \overline{x_n} y_n \,. \end{aligned} \tag{4.2}$$

Then we note that Φ is conjugate-linear due to the linearity of L_x, that is, for $x, z \in \ell_\infty$ and $\alpha \in \mathbb{C}$,

$$\Phi(x + z) = L_{x+z} = L_x + L_z = \Phi(x) + \Phi(z)\,, \tag{4.3}$$

$$\text{and} \quad \Phi(\alpha x) = L_{\alpha x} = \bar{\alpha} L_x = \bar{\alpha}\Phi(x)\,. \tag{4.4}$$

We will first show that Φ is an isometry. Firstly, note that $\|\Phi(x)\|_{\ell_1^*}$ is bounded above since Hölder's inequality tells us for $x \in \ell_\infty$ and $y \in \ell_1$,

$$\|L_x(y)\|_{\mathbb{C}} = \left| \sum_{n=1}^\infty \overline{x_n} y_n \right| \leq \sum_{n=1}^\infty |\overline{x_n} y_n| = \|xy\|_1 \leq \|x\|_\infty \|y\|_1\,, \tag{4.5}$$

which shows that

$$\|\Phi(x)\|_{\ell_1^*} = \|L_x\|_{\ell_1^*} = \sup_{y \in \ell_1 \setminus \{0\}} \frac{\|L_x(y)\|_{\mathbb{C}}}{\|y\|_1} \leq \|x\|_\infty\,, \tag{4.6}$$

which tells us that Φ is a bounded linear map since we at least have $\|\Phi\| \leq 1$. We now want to show that $\|\Phi(x)\|_{\ell_1^*}$ is also bounded below by $\|x\|_\infty$ - which reduces to attempting to show that for all $y \in \ell_1$ we have

$$\|\Phi(x)\|_{\ell_1^*} \geq \frac{\|L_x(y)\|_{\mathbb{C}}}{\|y\|_1} \geq \|x\|_\infty\,.$$

Without loss of generality assume $x \neq 0$. We can then define the sequence $y = (x_n/|x_n|)e_j$ for some $j \in \mathbb{N}$, where e_j is the standard basis vector with a 1 in the j position and 0 everywhere else (and also assume that $x_j \neq 0$). Then we have

$$\|y\|_1 = \sum_{n=1}^n \left| \frac{x_n}{|x_n|} e_j \right| = 1\,, \tag{4.7}$$

which then gives us

$$\|L_x(y)\|_{\mathbb{C}} = \left| \sum_{n=1}^\infty \overline{x_n} \frac{x_n}{|x_n|} e_j \right| = |x_j|\,. \tag{4.8}$$

Using these two facts we have

$$\|\Phi(x)\|_{\ell_1^*} \geq \frac{\|L_x(y)\|_{\mathbb{C}}}{\|y\|_1} = \frac{|x_j|}{1} = |x_j|\,, \tag{4.9}$$

but since this must be true for all $j \in \mathbb{N}$, we conclude that

$$\|\Phi(x)\|_{\ell_1^*} \geq \sup_{j\in\mathbb{N}} |x_j| = \|x\|_\infty\,. \tag{4.10}$$

Combining this with (4.6) gives us the desired isometry, namely for all $x \in \ell_\infty$,

$$\|\Phi(x)\|_{\ell_1^*} = \|x\|_\infty\,. \tag{4.11}$$

We now aim to show surjectivity, that is, for any $\lambda \in \ell_1^*$, there exists a $x \in \ell_\infty$ such that $L_x = \lambda$, that is, $L_x(y) = \lambda(y)$ for all $y \in \ell_1$. Let $\lambda \in \ell_1^*$ be fixed. Consider the sequence $e_i \in \ell_1$ defined in the standard way. Then to get the desired equality, we have

$$L_x(e_i) = \sum_{n=1}^{\infty} \overline{x_n} e_i = \overline{x_i} \quad \text{so we need} \quad \overline{\lambda(e_i)} = x_i\,, \tag{4.12}$$

which leads us to define our sequence

$$x = \{\overline{\lambda(e_n)}\}_{n=0}^{\infty}\,. \tag{4.13}$$

We then check that $x \in \ell_\infty$. For any $i \in \mathbb{N}$ we have

$$|\lambda(e_i)| \leq \|\lambda\|\|e_i\|_1 = \|\lambda\| < \infty \tag{4.14}$$

which we know is finite since λ is in the dual space, i.e. is a bounded linear operator. This tells us that for all $n \in \mathbb{N}$ we have

$$|x_n| \leq \|\lambda\| \quad \text{so} \quad \|x\|_\infty = \sup_{n\in\mathbb{N}} |x_n| \leq \|\lambda\| < \infty\,, \tag{4.15}$$

so we have $x \in \ell_\infty$. To conclude that $L_x(y) = \lambda(y)$ for all $y \in \ell_1$, we note that both L_x and λ are both linear, and with any finite sequence $y \in S$ defined in (2.1), we have

$$\begin{aligned} L_x(y) &= L_x(y_1 e_1 + \cdots + y_n e_n) \\ &= y_1 L_x(e_1) + \cdots + y_n L_x(e_n) \\ &= y_1 \lambda(e_1) + \cdots + y_n \lambda(e_n) \\ &= \lambda(y)\,, \end{aligned} \tag{4.16}$$

but since the set of finite sequences is dense in ℓ_1, and L_x and λ agree on a dense subset from the above calculation, we conclude that $L_x(y) = \lambda(y)$ for all $y \in \ell_1$ and so Φ is surjective.

Injectivity is clear since if $L_x(z) = L_x(z)$ for all $z \in \ell_1$ then

$$\sum_{n=1}^{\infty} \overline{x_n} z_i = \sum_{n=1}^{\infty} \overline{y_n} z_i \quad \text{so} \quad \sum_{n=1}^{\infty} (\overline{x_n - y_n}) z_i = 0 \quad \text{so } x = y\,. \tag{4.17}$$

Therefore, Φ induces the isomorphism of Banach spaces $\ell_\infty \cong \ell_1^*$. $\square$

To show that $\ell_\infty^* \not\cong \ell_1$, we will show that in this case $\Phi : \ell_1 \to \ell_\infty^*$ is not surjective by appealing to the Hahn-Banach theorem. That is, there exist functionals $\Lambda \in \ell_\infty^*$ that are not of the form L_x.

Consider the subspace $c \subset \ell_\infty$ of convergent sequences. Then consider $\lambda \in c^*$ defined as

$$\lambda(x) = \lim_{n\to\infty} x_n \quad \text{where} \quad |\lambda(x)| = |\lim_{n\to\infty} x_n| \leq \sup_{n\in\mathbb{N}} |x_n| = \|x\|_\infty \,, \tag{4.18}$$

where we know that $\|x\|_\infty$ exists since x is convergent. So we see that λ is a well defined bounded linear functional, hence is in c^*. By the Hahn-Banach theorem, this means we can find an extension $\Lambda \in \ell_\infty^*$ extending λ to ℓ_∞^* and satisfying $\|\Lambda\|_{\ell_\infty^*} = \|\lambda\|_{c^*}$. Suppose Λ was of the form L_x for $x \in \ell_1$. Then for $e_j \in c$ we would have

$$L_x(e_j) = \sum_{n=1}^{\infty} x_n e_j = x_j \,, \tag{4.19}$$

but since Λ must agree with λ on c, we have

$$\Lambda(e_j) = \lambda(e_j) = \lim_{n\to\infty} (\dots, 0, 1, 0, \dots) = 0 \,, \tag{4.20}$$

which implies that we must have

$$\Lambda(e_j) = 0 = x_j = L_x(e_j) \quad \text{for all } j \in \mathbb{N} \,, \tag{4.21}$$

meaning that Λ must be the 0 function since this is true for all x and j. But clearly if we choose the constant sequence $y = \{k\}_{n=1}^{\infty} \in c$ for some non-zero $k \in \mathbb{C}$, then

$$\Lambda(y) = \lambda(y) = k \,, \tag{4.22}$$

so Λ cannot be the 0 function. Hence we arrive at a contradiction and so we conclude that Λ cannot be of the form L_x - that is, there are functionals $\Lambda \in \ell_\infty^*$ such that there is no $x \in \ell_1$ that gives us $\Phi(x) = L_x = \Lambda$. Therefore Φ is not surjective and so ℓ_1 is *not* isomorphic to ℓ_∞. $\square$

Indeed, we proved in lectures that $\ell_\infty^* \cong c_0$. It is also worth noting that there is a useful theorem proved in Reed and Simon that says if X^* of a Banach space X is separable, then X is also separable. In question 2 we showed that ℓ_1 is separable but ℓ_∞ was not, which means that ℓ_1 could not be the dual of ℓ_∞.