Functional Analysis Assignment 1

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Q1. Metric on space of Cauchy Sequences

Part a)

Let (X, d) be a metric space and define

$$X' := \{\text{Cauchy sequences } \{x_n\}_{n=1}^{\infty} \text{ in } (X, d)\}$$

$$d' : X' \to \mathbb{R}, \;\; d'(\{x_n\}_{n=1}^{\infty}, \{y_n\}_{n=1}^{\infty}) := \lim_{n\to\infty} d(x_n, y_n)$$

We claim that d' is *not* a metric on X'. We will show that for $x, y \in X'$, with $x = \{x_n\}_{n=1}^{\infty}$ and $y = \{y_n\}_{n=1}^{\infty}$, that $d'(x, y) = 0 \;\not\Longrightarrow\; x = y$.

Consider the the metric space (X, d) with $X = \mathbb{R}$ and $d(a, b) = |a - b|$. Suppose $x_n = e^{-n}$ and $y_n = -e^{-n}$, which are both clearly convergent, hence Cauchy sequences, so $x, y \in X'$. Then

$$\begin{aligned} d'(x, y) &= \lim_{n\to\infty} d(x_n, y_n) \\ &= \lim_{n\to\infty} |e^{-n} - (-e^{-n})| \\ &= \lim_{n\to\infty} |2e^{-n}| \\ &= 0 \end{aligned}$$

However, clearly we see that $x \neq y$ as sequences. Thus, $d'(x, y) = 0 \;\not\Longrightarrow\; x = y$ and so we conclude that d' is *not* a metric on X' as claimed. □

Part b)

Let $\tilde{X} = X'/\sim$, where $\sim$ is the equivalence relation $\{x_n\}_{n=1}^{\infty} \sim \{y_n\}_{n=1}^{\infty}$ if and only if $d(x_n, y_n) \to 0$. Let $\tilde{d} : \tilde{X} \to \mathbb{R}$ be defined as $\tilde{d}([\{x_n\}_{n=1}^{\infty}], [\{y_n\}_{n=1}^{\infty}]) = d'(\{x_n\}_{n=1}^{\infty}, \{y_n\}_{n=1}^{\infty})$. We will first show that $\tilde{d}$ is a metric function, and then show that it is well defined.

$\underline{\tilde{d}}$ **is a metric function**

Let $x, y, z \in \tilde{X}$ be defined as in part a). Most properties of $\tilde{d}$ are derived from the fact that we inherit the well defined metric function d on X.

(i) $\tilde{d}([x],[y]) \geq 0$

Since $d(x_n, y_n) \geq 0$, clearly $\tilde{d}([x],[y]) = \lim_{n\to\infty} d(x_n, y_n) \geq 0$.

(ii) $\tilde{d}([x],[y]) = 0 \iff [x] = [y]$

(a) $\implies$ Assume $\tilde{d}([x],[y]) = 0$. Then $\lim_{n\to\infty} d(x_n, y_n) = 0$. By definition, this means $[x] = [y]$.

(b) $\impliedby$ Assume $[x] = [y]$. Then $\tilde{d}([x],[y]) = \tilde{d}([x],[x]) = \lim_{n\to\infty} d(x_n, x_n) = 0$ since d is a well defined metric.

(iii) $\tilde{d}([x],[y]) = \tilde{d}([y],[x])$

$$\tilde{d}([x],[y]) = \lim_{n\to\infty} d(x_n, y_n) = \lim_{n\to\infty} d(y_n, x_n) = \tilde{d}([y],[x])$$

(iv) $\tilde{d}([x],[z]) \leq \tilde{d}([x],[y]) + \tilde{d}([y],[z])$

$$\begin{aligned}\tilde{d}([x],[z]) &= \lim_{n\to\infty} (d(x_n, z_n)) \\ &\leq \lim_{n\to\infty} (d(x_n, y_n) + d(y_n, z_n)) \\ &= \lim_{n\to\infty} d(x_n, y_n) + \lim_{n\to\infty} d(y_n, z_n) \\ &= \tilde{d}([x],[y]) + \tilde{d}([y],[z])\end{aligned}$$

Hence we have shown that $\tilde{d}$ satisfies the necessary properties of a metric.

$\underline{\tilde{d}}$ **is well defined**

(i) $([x],[y]) \in \tilde{X} \times \tilde{X} \implies \tilde{d}([x],[y]) \in \mathbb{R}$

Take $([x],[y]) \in \tilde{X} \times \tilde{X}$. Since $d(x_n, y_n) \in \mathbb{R}$ we see that $\tilde{d}([x],[y]) = \lim_{n\to\infty} d(x_n, y_n) \in \mathbb{R}$ since $\mathbb{R}$ is complete.

(ii) $([x_1],[y_1]) = ([x_2],[y_2]) \implies \tilde{d}([x_1],[y_1]) = \tilde{d}([x_2],[y_2])$

Suppose $([x_1],[y_1]) = ([x_2],[y_2])$. Then

$$\begin{aligned}\tilde{d}([x_1],[y_1]) &= \lim_{n\to\infty} d((x_1)_n, (y_1)_n) \\ &= \lim_{n\to\infty} d((x_2)_n, (y_2)_n) \\ &= \tilde{d}([x_2],[y_2])\end{aligned}$$

since d is a well defined metric.

Hence we conclude that $\tilde{d}$ is a *well defined metric function* on $\tilde{X}$ as required. □

Q2. Unit ball in Hölder space

For $\alpha \in (0,1)$, the Hölder space of order α is denoted $C^\alpha[a,b]$, where

$$f \in C^\alpha[a,b] \iff \|f\|_{C^\alpha} = \sup_{x\in[a,b]} |f(x)| + \sup_{x,y\in[a,b]} \frac{|f(x)-f(y)|}{|x-y|^\alpha} < \infty$$

We then define the unit ball (open or closed does not matter - we choose closed) around 0 in $C^\alpha[a,b]$

$$B(0;1) = \{f \in C^\alpha[a,b] : \|f\|_{C^\alpha} \leq 1\}$$

To show that $B(0;1)$ is a pre-compact subset of $C^0[a,b]$, we can appeal to the Arzela-Ascoli theorem that says if $\mathcal{F}$ is a uniformly bounded and uniformly equicontinuous subset of $C^0[a,b]$, then $\mathcal{F}$ is pre-compact. By definition we have
$B(0;1) \subset C^\alpha[a,b] \subset C^0[a,b]$.

$B(0;1)$ is uniformly bounded if $\exists C > 0$ such that $\forall f \in B(0;1) \quad \|f\|_{C^\alpha} \leq C$. This is clear since by definition if $f \in B(0;1) \implies \|f\|_{C^\alpha} \leq 1$, so clearly $C = 1$ is suitable. Note that C is independent of f, causing the uniformity. Also, every f is also uniformly bounded with respect to the sup-norm since $\|f\|_\alpha \leq 1 \implies \sup_{x\in[a,b]} |f(x)| \leq 1 \implies |f(x)| \leq 1$ for all $x \in [a,b]$.

$B(0;1)$ is uniformly equi-continuous if

$$\forall \varepsilon > 0 \;\; \exists \delta > 0 \;\text{ s.t. }\; |x-y| < \delta \implies |f(x)-f(y)| < \varepsilon$$

where δ is *independent of* $x, y \in [a,b]$ *and* $f \in B(0;1)$. Let ε be fixed. Then $\forall f \in B(0;1)$ we have

$$\begin{aligned}
\|f\|_{C^\alpha} = \sup_{x\in[a,b]} |f(x)| + \sup_{x,y\in[a,b]} \frac{|f(x)-f(y)|}{|x-y|^\alpha} &\leq 1 \\
\implies \sup_{x,y\in[a,b]} \frac{|f(x)-f(y)|}{|x-y|^\alpha} \leq 1 - \sup_{x\in[a,b]} |f(x)| &\leq 1 \\
\implies \forall x, y \in [a,b] \quad \frac{|f(x)-f(y)|}{|x-y|^\alpha} &\leq 1 \\
\implies |f(x)-f(y)| &\leq |x-y|^\alpha
\end{aligned}$$

So if we choose $\delta = \varepsilon^{1/\alpha}$ (where α is fixed), then

$$|x-y| < \delta = \varepsilon^{1/\alpha} \implies |f(x)-f(y)| \leq |x-y|^\alpha < (\varepsilon^{1/\alpha})^\alpha = \varepsilon$$

where we used the fact that $(.)^\alpha$ is monotonically increasing for $0 < \alpha < 1$.
Since $\delta = \varepsilon^{1/\alpha}$ is independent of $x, y \in [a,b]$ and $f \in B(0;1)$, then we determine that $B(0;1)$ is uniformly equi-continuous.

Thus, by the Arzela-Ascoli Theorem, we see that $B(0;1)$ is a pre-compact subset of $C^0[a,b]$. $\square$

Q3. Inner Product Spaces

Part a)

Let $(V, (\ \cdot\ ,\ \cdot\ ))$ be an inner product space, $x, y \in V$. Define $\|x\|^2 = (x, x)$. Then

$$\begin{aligned} \|x+y\|^2 - \|x-y\|^2 &= (x+y, x+y) - (x-y, x-y) \\ &= (x,x) + (x,y) + (y,x) + (y,y) \\ &\quad - [(x,x) + (x,-y) + (-y,x) + (-y,-y)] \\ &= (x,y) + \overline{(x,y)} - [-(x,y) - (y,x)] \\ &= 2(x,y) + 2\overline{(x,y)} \\ &= 4\,\mathrm{Re}(x,y) \end{aligned}$$

Where we appealed to the fact that $(-x, y) = (x, -y) = -(x, y)$ and $(y, x) = \overline{(x, y)}$. Also, $\mathrm{Re}(z) = \frac{z+\bar{z}}{2}$. Similarly,

$$\begin{aligned} \|x+iy\|^2 - \|x-iy\|^2 &= (x+iy, x+iy) - (x-iy, x-iy) \\ &= (x,x) + (x,iy) + (iy,x) + (y,y) \\ &\quad - [(x,x) + (x,-iy) + (-iy,x) + (-iy,-iy)] \\ &= i(x,y) - i\overline{(x,y)} - [-i(x,y) + i\overline{(x,y)}] \\ &= 2i(x,y) - 2i\overline{(x,y)} \\ &= 4i\,\mathrm{Im}(x,y) \end{aligned}$$

Where we appealed to the fact that $(x, iy) = (-ix, y) = i(x, y)$ and $\mathrm{Im}(z) = \frac{z-\bar{z}}{2i}$. Hence,

$$\begin{aligned} \frac{1}{4}((\|x+y\|^2 - \|x-y\|^2) - i(\|x+iy\|^2 - \|x-iy\|^2)) &= \frac{1}{4}(4\,\mathrm{Re}(x,y) + 4\,\mathrm{Im}(x,y)) \\ &= (x,y) \end{aligned}$$

This proves the polarisation identity holds for a given inner product space V. □

Part b)

Let $(V, \|.\|)$ be a normed linear space (NLS). A NLS satisfies the parallelogram law if it obeys the following identity $\forall x, y \in V$

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|x\|^2$$

(i) **Inner product space $\implies$ parallelogram law**

Suppose V is also an inner product space (IPS) with the inner product defined in the standard way. Then we have

$$\begin{aligned} \|x+y\|^2 + \|x-y\|^2 &= (x+y, x+y) + (x-y, x-y) \\ &= (x,x) + (x,y) + (y,x) + (y,y) \\ &\quad + (x,x) - (x,y) - (y,x) + (-y,-y) \\ &= 2(x,x) + 2(y,y) \\ &= 2\|x\|^2 + 2\|y\|^2 \end{aligned}$$

and so the IPS V obeys the parallelogram law as required.

(ii) **NLS with parallelogram law $\implies$ IPS**

Now suppose that V satisfies the parallelogram law. We wish to show that the inner product defined by the polarisation identity is indeed a well defined inner product satisfying all of the necessary conditions.

Let $x, y, z \in V$ and $\alpha \in \mathbb{C}$.

(i) $(x, y) = \overline{(y, x)}$

$$\begin{aligned}
\overline{(y,x)} &= \overline{\frac{1}{4}\left((\|y+x\|^2 - \|y-x\|^2) - i(\|y+ix\|^2 - \|y-ix\|^2)\right)} \\
&= \frac{1}{4}\left((\|x+y\|^2 - \|x-y\|^2) - i(\|y-ix\|^2 - \|y+ix\|^2)\right) \\
&= \frac{1}{4}\left((\|x+y\|^2 - \|x-y\|^2) - i\left(|i|\,\|y-ix\|^2 - |-i|\,\|y+ix\|^2\right)\right) \\
&= \frac{1}{4}\left((\|x+y\|^2 - \|x-y\|^2) - i\left(\|(i)(y-ix)\|^2 - \|(-i)(y+ix)\|^2\right)\right) \\
&= \frac{1}{4}\left((\|x+y\|^2 - \|x-y\|^2) - i\left(\|x+iy\|^2 - \|x-iy\|^2\right)\right) \\
&= (x,y)
\end{aligned}$$

Where we used the fact that $|-i| = |i| = 1$ and $|\alpha|^2\,\|x\|^2 = \|\alpha x\|^2$.

(ii) $(x, x) \geq 0$ and $(x, x) = 0 \implies x = 0$

$$\begin{aligned}
(x,x) &= \frac{1}{4}\left((\|x+x\|^2 - \|x-x\|^2) - i\left(\|x+ix\|^2 - \|x-ix\|^2\right)\right) \\
&= \frac{1}{4}\left(4\|x\|^2 - i\left(|1+i|^2\,\|x\|^2 - |1-i|^2\,\|x\|^2\right)\right) \\
&= \frac{1}{4}\left(4\|x\|^2 - i(2\|x\|^2 - 2\|x\|^2)\right) \\
&= \|x\|^2 \geq 0
\end{aligned}$$

Where the last line is clear from properties of a norm.
Now assume that $(x, x) = 0$. Since $(x, x) = \|x\|^2$ as shown above, this implies that $\|x\|^2 = 0$ but since $\|.\|$ is a norm, this is only true when $x = 0$. Hence $(x, x) = 0 \implies x = 0$.

(iii) $(x, iy) = i(x, y)$ (sub-property - full property discussed later)

$$\begin{aligned}
(x,iy) &= \frac{1}{4}\left((\|x+iy\|^2 - \|x-iy\|^2) - i(\|x+i(iy)\|^2 - \|x-i(iy)\|^2)\right) \\
&= \frac{1}{4}\left(i(\|x+y\|^2 - \|x-y)\|^2) + (\|x+iy\|^2 - \|x-iy\|^2)\right) \\
&= (i)\frac{1}{4}\left((\|x+y\|^2 - \|x-y)\|^2) - i(\|x+iy\|^2 - \|x-iy\|^2)\right) \\
&= i(x,y)
\end{aligned}$$

(iv) $(x+y,z) = (x,z) + (y,z)$

We first look at $(x+y,z)$ to see what objects we are interested in studying.

$$\begin{aligned}(x+y,z) = \frac{1}{4}\big((&\|(x+y)+z\|^2 - \|(x+y)-z\|^2) \\ &-i\,(\|(x+y)+iz\|^2 - \|(x+y)-iz\|^2\big)\end{aligned}$$

Using the parallelogram law, we see

$$\|(x+z)+y\|^2 = 2\|x+z\|^2 + 2\|y\|^2 - \|x+z-y\|^2 \tag{3.1}$$
$$\|(y+z)+x\|^2 = 2\|y+z\|^2 + 2\|x\|^2 - \|y+z-x\|^2 \tag{3.2}$$

Rearranging, we see that

$$\begin{aligned}\implies \|x+y+z\|^2 &= \frac{1}{2}\big[(2\|x+z\|^2 + 2\|y\|^2 - \|x+z-y\|^2) \\ &\quad + (\ 2\|y+z\|^2 + 2\|x\|^2 - \|y+z-x\|^2)\big] \\ &= \|x\|^2 + \|y\|^2 + \|x+z\|^2 + \|y+z\|^2 \\ &\quad - \frac{1}{2}\|x+z-y\|^2 - \frac{1}{2}\|y+z-x\|^2\end{aligned}$$

We can then make the substitution sending $z \mapsto -z$ to get the following

$$\begin{aligned}\|x+y-z\|^2 &= \|x\|^2 + \|y\|^2 + \|x-z\|^2 + \|y-z\|^2 \\ &\quad - \frac{1}{2}\|x-y-z\|^2 - \frac{1}{2}\|y-x-z\|^2 \\ &= \|x\|^2 + \|y\|^2 + \|x-z\|^2 + \|y-z\|^2 \\ &\quad - \frac{1}{2}\|-(x-y-z)\|^2 - \frac{1}{2}\|-(y-x-z)\|^2 \\ &= \|x\|^2 + \|y\|^2 + \|x-z\|^2 + \|y-z\|^2 \\ &\quad - \frac{1}{2}\|y+z-x\|^2 - \frac{1}{2}\|x+z-y\|^2\end{aligned}$$

Comparing terms, we get

$$\begin{aligned}\mathrm{Re}(x+y,z) &= \frac{1}{4}\left(\|x+y+z\|^2 - \|x+y-z\|^2\right) \\ &= \frac{1}{4}\left((\|x+z\|^2 - \|x-z\|^2) + (\|y+z\|^2 - \|y-z\|^2)\right) \\ &= \mathrm{Re}(x,z) + \mathrm{Re}(y,z)\end{aligned}$$

Similarly for the imaginary part, and sending $z \mapsto iz$ in our previous identities,

$$\begin{aligned}\mathrm{Im}(x+y,z) &= -\frac{1}{4}\left(\|x+y+iz\|^2 - \|x+y-iz\|^2\right) \\ &= -\frac{1}{4}\left((\|x+iz\|^2 - \|x-iz\|^2) + (\|y+iz\|^2 - \|y-iz\|^2)\right) \\ &= -\mathrm{Re}(x,iz) - \mathrm{Re}(y,iz) \\ &= -\mathrm{Re}(i(x,z)) - \mathrm{Re}(i(y,z)) \\ &= \mathrm{Im}(x,z) + \mathrm{Im}(y,z)\end{aligned}$$

Hence, since $\text{Re}(x+y,z) = \text{Re}(x,z) + \text{Re}(y,z)$ and $\text{Im}(x+y,z) = \text{Im}(x,z) + \text{Im}(y,z)$, by the uniqueness of complex numbers we conclude that $(x+y,z) = (x,z) + (y,z)$ as required. We also notice linearity in the second term

$$(x, y+z) = \overline{(y+z,x)} = \overline{(y,x)+(z,x)} = \overline{(y,x)} + \overline{(z,x)} = (x,y) + (x,z)$$

(v) $(x, \alpha y) = \alpha(x,y)$

We have already shown this is the case for $(x, iy) = i(x,y)$. It is trivial to show that $(x,-y) = -(x,y)$ and that $(x,0) = 0$. By induction, from the linearity of property iv), we can show that for $n \in \mathbb{N}$, $(x, ny) = (x,y) + \cdots + (x,y) = n(x,y)$. Combining this with the aforementioned properties, we get that for $n \in \mathbb{Z}$ we have $(x, ny) = n(x,y)$. We now want to consider the case of $\beta \in \mathbb{Q}$. Consider $\beta = m/n \in \mathbb{Q}$ for $m, n \in \mathbb{Z}$ $(n \neq 0)$. Then we see, using the properties for $m, n \in \mathbb{Z}$ as mentioned above,

$$\frac{1}{\beta}(x, \beta y) = \frac{n}{m}(x, \frac{m}{n}y) = \frac{nm}{m}(x, \frac{1}{n}y) = n(x, \frac{1}{n}y) = (x, n\frac{1}{n}y) = (x,y)$$
$$\implies (x, \beta y) = \beta(x,y)$$

Which shows that the property holds for $\beta \in \mathbb{Q}$. To extend this result to $\mathbb{R}$, we define $\phi : \mathbb{R} \to \mathbb{C}, \ \ \phi(\alpha) = (x, \alpha y)$ and $\psi : \mathbb{R} \to \mathbb{C}, \ \ \psi(\alpha) = \alpha(x,y)$. Both ϕ and ψ are continuous functions due to the continuity of the norm that induces the inner product. We showed above that $\phi|_{\mathbb{Q}} = \psi|_{\mathbb{Q}}$, and then we can use the fact that if two continuous functions agree on a dense subset of their preimage (i.e. $\mathbb{Q} \subset \mathbb{R}$) then they agree everywhere. Thus, we have $\phi = \psi$. Extending this to $\mathbb{C}$ with the property $(x, iy) = i(x,y)$ and linearity in the second term, we see that $(x, \alpha y) = \alpha(x,y) \ \forall \alpha \in \mathbb{C}$ as required.

Thus, since (x,y) induced by $\|.\|$ with the parallelogram law obeys all necessary conditions for an inner product, we conclude that a normed linear space is an inner product space if and only if the norm satisfies the parallelogram law. □

Part c)

Let $(C^0[a,b], \|.\|_\infty)$ be the normed linear space of continuous functions, where for $f \in C^0[a,b]$ we define $\|f\|_\infty = \max_{x\in[a,b]}\big|f(x)\big|$. We will show by use of a counterexample that this norm does *not* obey the parallelogram law for specific f and g.

Let $f, g \in C^0[a,b]$ be defined as $f(x) = x$ and $g(x) = 1 - x$ on the interval $[0,1]$. Then

$$\begin{aligned}
\|f+g\|_\infty^2 + \|f-g\|_\infty^2 &= \left(\max_{x\in[0,1]}\big|f(x)+g(x)\big|\right)^2 + \left(\max_{x\in[0,1]}\big|f(x)-g(x)\big|\right)^2 \\
&= \left(\max_{x\in[0,1]}|1|\right)^2 + \left(\max_{x\in[0,1]}|1-2x|\right)^2 \\
&= 1^2 + 1^2 = 2
\end{aligned}$$

But,

$$\begin{aligned}
2\|f\|_\infty^2 + 2\|g\|_\infty^2 &= 2\left(\max_{x\in[0,1]}\big|f(x)\big|\right)^2 + 2\left(\max_{x\in[0,1]}\big|g(x)\big|\right)^2 \\
&= 2\left(\max_{x\in[0,1]}|x|\right)^2 + 2\left(\max_{x\in[0,1]}|1-x|\right)^2 \\
&= 2(1^2) + 2(1^2) = 4
\end{aligned}$$

Hence, we observe that $\|f+g\|_\infty^2 + \|f-g\|_\infty^2 \neq 2\|f\|_\infty^2 + 2\|g\|_\infty^2$ and so the parallelogram law does not hold for this case. Therefore, using part b), we deduce that $(C^0[a,b], \|.\|_\infty)$ is not an inner product space. □

Q4. Closed subspaces of $C^0[a,b]$ are not as nice

We have proven that for a Hilbert space $\mathcal{H}$ with $\mathcal{M} \subset \mathcal{H}$ a closed subspace, then $\forall v \in \mathcal{H}$ there is a unique $w \in \mathcal{M}$ satisfying $\|w - v\| = \inf_{w' \in \mathcal{M}} \|w' - v\|$. We will construct a counter example for the normed linear space $(C^0[a,b], \|.\|_\infty)$. Without loss of generality, assume the interval $[a,b]$ is $[0,1]$ for ease.

Consider the subspace $\mathcal{X} \subset C^0[0,1]$ defined by:

$$\mathcal{X} := \{g \in C^0[0,1] : g(0) = 0\}$$

The fact that $\mathcal{X}$ is a subspace is clear - the fact that it is closed in the topological sense deserves some attention. Consider the functional $T : C^0[0,1] \to \mathbb{R}$ defined by $T(g) = g(0)$. It is clear that T is a linear functional. We also see that T is bounded since $|T(g)| = |g(0)| \leq \|g\|_\infty$ (since $\|g\|_\infty$ is finite $\forall g \in C^0[a,b]$). Hence, by the lemma in class, this implies that T is a continuous linear functional. We know that under continuous functions, the pre-image of a closed subset is closed. Hence, $T^{-1}(\{0\}) = \mathcal{X}$ is closed in the topological sense. Thus $\mathcal{X}^0[0,1]$ is a closed subspace.

Now consider the function $f \in C^0[0,1]$ with $f(x) = 1$. We will show that there are multiple $g \in \mathcal{X}$ that infimise the distance to f. We see that since $f(0) = 1$ and $(\forall g \in \mathcal{X})$ $g(0) = 0$, we have $|g(0) - f(0)| = |0 - 1| = 1$. This tells us that

$$\inf_{g' \in \mathcal{X}} \|g' - f\|_\infty = \inf_{g' \in \mathcal{X}} \left(\max_{x\in[0,1]} |g'(x) - f(x)|\right) \geq 1$$

Consider the functions $g_1, g_2 \in \mathcal{X}$ defined by $g_1(x) = x$ and $g_2(x) = 2x$. Then

$$\begin{aligned}
\|g_1 - f\|_\infty &= \max_{x\in[0,1]} |x - 1| = 1 \\
\|g_2 - f\|_\infty &= \max_{x\in[0,1]} |2x - 1| = 1
\end{aligned}$$

Since we have shown that $\inf_{g' \in \mathcal{X}} \|g' - f\|_\infty \geq 1$, and we have found two distinct $g_1, g_2 \in \mathcal{X}$ that both infimise the distance to the function $f \in C^0[0,1]$, we conclude that $(C^0[a,b], \|.\|_\infty)$ does not have this same 'unique closest element' property that we observed for a closed subspace of $\mathcal{H}$. □

Q5. ℓ^p space is Banach

Let $(\ell^p, \|.\|_p)$ (with $1 < p \leq \infty$) denote the normed linear space of sequences that converge with respect to the p-norm, that is, for a sequence of complex numbers $x = \{x_i\}_{i=1}^{\infty} \in \ell^p$, define

$$\|x\|_p := \left(\sum_{i=1}^{\infty} |x_i|^p \right)^{1/p}, \quad 1 < p < \infty, \qquad \|x\|_\infty := \sup_{i \in \mathbb{N}} |x_i|$$

The fact that both of these definitions define a normed linear space is trivial given that we may use, without proof, the Minkowski inequality to verify the triangle inequality (i.e. $\forall x, y \in \ell^p, \|x+y\|_p \leq \|x\|_p + \|y\|_p$). Proving completeness is clearly non-trivial, so we divide into the two separate cases. By definition, ℓ^p is complete if $\exists X \in \ell^p \quad \text{s.t} \quad \lim_{n\to\infty} \|X^{(n)} - X\|_p = 0$.

$\underline{1 < \mathbf{p} < \infty}$

Consider a sequence of elements in ℓ^p denoted by $X^{(n)} = \{x^{(n)}\}_{n=1}^{\infty}$ where $x^{(n)} = \{x_i^{(n)}\}_{i=1}^{\infty} \in \ell^p$. Let $X^{(n)}$ be a Cauchy sequence - that is,

$$\forall \varepsilon > 0 \quad \exists N \in \mathbb{N} \quad \text{s.t.} \quad \forall n, m \geq N \quad d_p(X^{(n)}, X^{(m)}) < \varepsilon$$

where we define

$$d_p(X^{(n)}, X^{(m)}) = \|X^{(n)} - X^{(m)}\|_p = \left(\sum_{i=1}^{\infty} |x_i^{(n)} - x_i^{(m)}|^p \right)^{1/p}$$

We wish to show that the sequence $X = \{x_i\}_{i=1}^{\infty}$ is an element of ℓ^p (i.e. $\|X\|_p$ is finite) and that $\lim_{n\to\infty} \|X^{(n)} - X\|_p = 0$. Clearly, the natural choice for X is $X = \{\lim_{n\to\infty} x_i^{(n)}\}_{i=1}^{\infty}$.

We first notice that for a fixed $j \in \mathbb{N}$, the sequence $X_j^{(n)} = \{x_j^{(n)}\}_{n=1}^{\infty} \subset \mathbb{C}$ is Cauchy since $\forall n, m \geq N$

$$\|X_j^{(n)} - X_j^{(m)}\|_{\mathbb{R}}^p = |x_j^{(n)} - x_j^{(m)}|^p \leq \sum_{j=1}^{\infty} |x_j^{(n)} - x_j^{(m)}|^p = \|X^{(n)} - X^{(m)}\|_p^p < \varepsilon^p$$

We can then use the fact that for fixed $j, n \in \mathbb{N}$ we have $x_j^{(n)} \in \mathbb{C}$. Since $\mathbb{C}$ is complete, we see that our Cauchy sequence $X_j^{(n)}$ must converge to an element $x_j \in \mathbb{C}$. Define this as $\lim_{n\to\infty} X_j^{(n)} = x_j$.

For the finite sum with a fixed $K \in \mathbb{N}$, we have $\forall m, n \geq N$ that

$$\sum_{j=1}^{K} |x_j^{(n)} - x_j^{(m)}|^p \leq \sum_{j=1}^{\infty} |x_j^{(n)} - x_j^{(m)}|^p = \|X^{(n)} - X^{(m)}\|_p^p < \varepsilon^p$$

Since we are now dealing with a finite sum and $|.|$ is a continuous function, and using basic properties of limits on inequalities (i.e. if $\forall n \;\; a_n < b_n \implies \lim_{n\to\infty} a_n \leq \lim_{n\to\infty} b_n$), we see that $\forall n > N$ we can move the limit inside the sum as follows

$$\begin{aligned} \lim_{m\to\infty} \sum_{j=1}^{K} |x_j^{(n)} - x_j^{(m)}|^p &\leq \lim_{m\to\infty} \varepsilon^p \\ \implies \sum_{j=1}^{K} |x_j^{(n)} - \lim_{m\to\infty} x_j^{(m)}|^p &\leq \varepsilon^p \\ \implies \sum_{j=1}^{K} |x_j^{(n)} - x_j|^p &\leq \varepsilon^p \end{aligned} \tag{5.1}$$

We now appeal to the Minkowski inequality. Though this statement is relevant for an infinite sum, we may regard our finite sum over $j = 1, \dots, K$ as being an infinite sum over a sequence that is identically 0 $\forall j > K$, hence making it valid to use this inequality. Thus $\forall n > N$ we have

$$\begin{aligned} \left(\sum_{j=1}^{K} |x_j|^p\right)^{1/p} &= \left(\sum_{j=1}^{K} |x_j - x_j^{(n)} + x_j^{(n)}|^p\right)^{1/p} \\ &\leq \left(\sum_{j=1}^{K} |x_j - x_j^{(n)}|^p\right)^{1/p} + \left(\sum_{j=1}^{K} |x_j^{(n)}|^p\right)^{1/p} \\ &\leq \varepsilon + \left(\sum_{j=1}^{K} |x_j^{(n)}|^p\right)^{1/p} \end{aligned}$$

If we now let $K \to \infty$, again appealing to limit inequality properties from before, we arrive at the crucial inequality that tells us that $X = \{x_j\}_{j=1}^{\infty}$ is in ℓ^p since:

$$\begin{aligned} \lim_{K\to\infty} \left(\sum_{j=1}^{K} |x_j|^p\right)^{1/p} &\leq \lim_{K\to\infty} \left[\varepsilon + \left(\sum_{j=1}^{K} |x_j^{(n)}|^p\right)^{1/p}\right] \\ \implies \left(\sum_{j=1}^{\infty} |x_j|^p\right)^{1/p} &\leq \varepsilon + \left(\sum_{j=1}^{\infty} |x_j^{(n)}|^p\right)^{1/p} \\ \therefore \quad \|X\|_p &\leq \varepsilon + \|X^{(n)}\|_p \end{aligned}$$

Since this statement must be true for any fixed $\varepsilon > 0$ and any fixed $n > N$, and since we know that $\|X^{(n)}\|_p$ is finite since for fixed n, $X^{(n)} = \{x_j^{(n)}\}_{j=1}^{\infty} \in \ell^p$, this tells us that $\|X\|_p$ itself must be finite, hence $X \in \ell^p$.

Now we just need to show that $\lim_{n\to\infty} \|X^{(n)} - X\|_p = 0$. But this is clear since if we take $K \to \infty$ in (5.1), we get that for $n > N$

$$\|X^{(n)} - X\|_p^p = \sum_{j=1}^{\infty} |x_j^{(n)} - x_j|^p \leq \varepsilon^p$$

Thus since we have this for any ε, we have shown that $\lim_{n\to\infty} \|X^{(n)} - X\|_p = 0$ as required. Thus, $X^{(n)} = \{x^{(n)}\}_{n=1}^{\infty} \subset \ell^p$ is a convergent sequence that converges to $X = \{x_j\}_{j=1}^{\infty} \in \ell^p$, hence ℓ^p is a complete normed linear space for $1 < p < \infty$. $\square$

<u>**$\mathbf{p = \infty}$**</u>

Once again consider a Cauchy sequence $X^{(n)}$ as before, this time with

$$d_\infty(X^{(n)}, X^{(m)}) = \|X^{(n)} - X^{(m)}\|_\infty = \sup_{i\in\mathbb{N}} |x_i^{(n)} - x_i^{(m)}| < \varepsilon$$

Again, fix a $j \in \mathbb{N}$ to see that $X_j^{(n)}$ is Cauchy since

$$\|X_j^{(n)} - X_j^{(m)}\|_\mathbb{R} = |x_j^{(n)} - x_j^{(m)}| \leq \sup_{i\in\mathbb{N}} |x_i^{(n)} - x_i^{(m)}| = \|X^{(n)} - X^{(m)}\|_\infty^p < \varepsilon$$

By the same argument as above ($\mathbb{C}$ is complete, etc.), we have $\lim\limits_{n\to\infty} X_j^{(n)} = x_j \in \mathbb{C}$.

We now appeal to the fact that $\|.\|_\infty$ is a continuous function and basic properties of sup to show that

$$\begin{aligned}
\lim_{m\to\infty} \sup_{i\in\mathbb{N}} |x_i^{(n)} - x_i^{(m)}| &\leq \lim_{m\to\infty} \varepsilon \\
\sup_{i\in\mathbb{N}} |x_i^{(n)} - \lim_{m\to\infty} x_i^{(m)}| &\leq \lim_{m\to\infty} \varepsilon \\
\implies \|X^{(n)} - X\|_\infty \leq \varepsilon
\end{aligned}$$

which shows us that $\lim_{n\to\infty} \|X^{(n)} - X\|_\infty = 0$. Hence we can also now see that

$$\begin{aligned}
\|X\|_\infty = \sup_{i\in\mathbb{N}} |x_i| &= \sup_{i\in\mathbb{N}} |x_i - x_i^{(n)} + x_i^{(n)}| \\
&\leq \sup_{i\in\mathbb{N}} \left(|x_i - x_i^{(n)}| + |x_i^{(n)}|\right) \\
&\leq \sup_{i\in\mathbb{N}} \left(|x_i^{(n)} - x_i|\right) + \sup_{i\in\mathbb{N}} \left(|x_i^{(n)}|\right) \\
&\leq \varepsilon + \|X^{(n)}\|
\end{aligned}$$

Again, this shows that $\|X\|_\infty$ is finite, hence $X \in \ell^p$. Thus we have shown that $(\ell^p, \|.\|_p)$ is a Banach space as required. $\square$